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# A torus-like black hole *

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**Abstract**

A charged static torus-like black hole solution is constructed from the plane-symmetric solution of the Einstein–Maxwell equations with negative cosmological constant. Each surface of the space–time at constant $t$ and $r$ has a toroidal topology so that the topology theorem of black holes does not apply to this case.

The black hole is one of the most fascinating structures predicted by general relativity. Many efforts have been made to understand and find it. In general relativity it has been found that there exists a family of black holes which are described by four parameters [1], mass $M$, angular momentum $J$, electric charge $Q$, and the cosmological constant $\Lambda$ (if there exists magnetic charge in nature, then a fifth parameter would be added). The topology of the space-like cross section of the event horizon of these solutions is $S^2$. Are there any black hole solutions of Einstein's field equations whose event horizons have other topologies? Many authors have discussed this problem [2,3]. The topology theorem of black holes has been proved, which, roughly speaking, claims that the topology of the two-dimensional event horizon of a black hole is $S^2$ provided that the space–time is asymptotically flat and certain reasonable conditions such as energy conditions hold. The purpose of the present Letter, however, is to show that there do *exist* torus-like black hole solutions of the Einstein–Maxwell equations with negative cosmological constant.

The Letter proceeds as follows: First, we will construct a torus-like solution of the Einstein–Maxwell equations. Then we will study the properties of the solution. We will determine the physical meaning of the integration constants in the solution, investigate the singularity and give the "Penrose diagram" for the solution. Finally, a brief conclusion and discussion will be given. Throughout the Letter, the units $c = G = 1$ are used.

Consider the metric

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2(d\varphi^2 + d\psi^2), \qquad (1)$$

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where $B(r)$ and $A(r)$ are strictly positive functions. Obviously, (1) admits three spacelike Killing vectors $\partial/\partial\varphi, \partial/\partial\psi$ and $\varphi\partial/\partial\psi - \psi\partial/\partial\varphi$ and a time-like Killing vector $\partial/\partial t$. This implies that (1) may describe the static plane-symmetric space–time. For metric (1), the general solution of the Einstein–Maxwell equations with cosmological constant $\Lambda$ is [4,5]

$$ds^2 = \left( -\tfrac{1}{3}\Lambda r^2 - \frac{a}{r} + \frac{b^2}{r^2} \right) dt^2 - \frac{1}{-\tfrac{1}{3}\Lambda r^2 - a/r + b^2/r^2} dr^2 - r^2(d\varphi^2 + d\psi^2), \tag{2}$$

where $a$ and $b$ are two integration constants to be determined. The metric (2) is of Petrov type D and tends to the de Sitter metric for positive $\Lambda$ and the anti-de Sitter metric for negative $\Lambda$ as $r \to +\infty$. In particular, when $\Lambda = 0$, (2) reduces to the Kar metric [6].

It is easy to construct a torus-like solution by identifying $\varphi = 0$ with $\varphi = 2\pi$ and $\psi = 0$ with $\psi = 2\pi$ in (2). In other words, (2) may describe a space-time with torus topology. Although the local geometry remains unchanged in the identification, the global properties of the torus-like space–time are different from those of a plane-symmetric space–time. In the following, we will show that (2) is a black hole solution when it represents a torus-like space–time.

First, we determine the physical meaning of the constants $a$ and $b$. It is easy to see from the Gauss theorem that $b = (4/\pi)^{1/2}Q$ where $Q$ is the electric charge on the torus, since the non-zero component of the electric field is $E_1 = (4\pi)^{-1/2}b/r^2$ [5]. Note that the surface used in the Gauss theorem is a torus surrounding $r = 0$ in the present case rather than a sphere. It can be shown that $a$ is related to the ADM mass of the torus. Some comments on the ADM mass should be made. The ADM mass is introduced to define the mass of an isolated system. In the usual cases, the manifolds possess $S^2 \times \mathcal{M}^2$ topologies with $\mathcal{M}^2$ some two-dimensional manifolds such as $\mathbb{R}^2$. Therefore, one considers the 3+1 decomposition of the manifold bounded by $\Sigma_1$, $\Sigma_2$, and $[t_1, t_2]\times$a two-dimensional, simply-connected surface, where $\Sigma_1$ and $\Sigma_2$ are the hypersurfaces at $t_1$ and $t_2$, and define the ADM mass of the isolated system from the boundary term for the two-dimensional, simply-connected surface in the limit of $r \to \infty$. In the present case, the topology of the space-time manifold is $S \times S \times \mathcal{M}^2$. In order to adapt to the symmetry of the space–time, one has to consider, in the 3+1 decomposition, the manifold bounded by a hypersurface with $S \times S \times [t_1, t_2]$ topology in addition to $\Sigma_1$ and $\Sigma_2$. Again, one may define a mass from the boundary term for the two-dimensional toroidal boundary in the same way as in the usual cases. This is what we call the ADM mass in the present Letter. The calculation following Ref. [7] shows the ADM mass $M = \frac{1}{2}\pi a$. Thus, the metric (2) may be rewritten as

$$ds^2 = \left( -\tfrac{1}{3}\Lambda r^2 - \frac{2M}{\pi r} + \frac{4Q^2}{\pi r^2} \right) dt^2 - \frac{1}{-\tfrac{1}{3}\Lambda r^2 - 2M/\pi r + 4Q^2/\pi r^2} dr^2 - r^2(d\varphi^2 + d\psi^2), \tag{3}$$

with $0 \leqslant \varphi \leqslant 2\pi$ (identified with 0) and $0 \leqslant \psi \leqslant 2\pi$ (identified with 0).

Next, we should analyze the singularity of (3). The metric (3) becomes singular for

$$g_{00} = -\tfrac{1}{3}\Lambda r^2 - \frac{2M}{\pi r} + \frac{4Q^2}{\pi r^2} = 0 \tag{4}$$

and at $r = 0$. For negative $\Lambda$, (4) has two positive solutions[4],

$$r_\pm = \tfrac{1}{2}\left\{ \Upsilon^{1/2} \pm \left[ -\Upsilon + 2\left( \Upsilon^2 + \frac{48}{\pi\Lambda}Q^2 \right)^{1/2} \right]^{1/2} \right\} \tag{5}$$

[4] When $\Lambda > 0$, there exists a single positive solution of (4) for any $M$ and $Q$. It is $r_h = \frac{1}{2}(\Upsilon^{1/2} + \{-\Upsilon + 2[\Upsilon^2 + (48/\pi\Lambda)Q^2]^{1/2}\}^{1/2})$, where $\Upsilon$ is given by (7). Incorporating the fact that (3) is not static for large $r$, it implies that the metric with positive $\Lambda$ does not describe a black hole space–time.

as long as

$$0 \leqslant Q^2 \leqslant \tfrac{3}{8}(3M^4/2\pi|\Lambda|)^{1/3}, \tag{6}$$

where

$$\Upsilon = \left\{ \frac{(6M)^2}{2\pi^2\Lambda^2} + \left[ \left( \frac{(6M)^2}{2\pi^2\Lambda^2} \right)^2 + \left( \frac{16}{\pi\Lambda} Q^2 \right)^2 \right]^{1/2} \right\}^{1/3} + \left\{ \frac{(6M)^2}{2\pi^2\Lambda^2} - \left[ \left( \frac{(6M)^2}{2\pi^2\Lambda^2} \right)^2 + \left( \frac{16}{\pi\Lambda} Q^2 \right)^2 \right]^{1/2} \right\}^{1/3}. \tag{7}$$

The singularities at $r_\pm$ may be removed by use of the retarded coordinate system, $(u, r, \theta, \varphi)$ or the advanced coordinate system, $(v, r, \theta, \varphi)$. The null coordinates, $u$ and $v$, are defined by

$$u = t - r^*, \qquad t + r^*, \tag{8}$$

where

$$r^* = \int \frac{\mathrm{d}r}{-\frac{1}{3}\Lambda r^2 - 2GM/\pi r + 4GQ^2/\pi r^2}. \tag{9}$$

When $Q^2 < \frac{3}{8}(3M^4/2\pi|\Lambda|)^{1/3}$, $r_+ \neq r_-$,

$$\begin{aligned} r^* = \frac{3}{\Lambda}\Bigg( &- \frac{1}{r_+ - r_-} \log\left|\frac{r - r_+}{r - r_-}\right| \\ &- \frac{(r_+ + r_-)^4 + 2(r_+^2 + r_-^2)^2 + 2r_+r_-(r_+^2 + r_-^2)}{[(r_+ + r_-)^2 + 2r_+^2][(r_+ + r_-)^2 + 2r_-^2][(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)]^{1/2}} \\ &\times \arctan \frac{2r + r_+ + r_-}{[(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)]^{1/2}} \\ &+ \frac{(r_+ + r_-)^2 + r_+^2}{(r_+ - r_-)[(r_+ + r_-)^2 + 2r_+^2]} \log|r - r_+| - \frac{(r_+ + r_-)^2 + r_-^2}{(r_+ - r_-)[(r_+ + r_-)^2 + 2r_-^2]} \log|r - r_-| \\ &+ \frac{(r_+ + r_-)^3}{4(r_+ + r_-)^4 + 2(r_+^2 + r_-^2)^2} \log|r^2 + (r_+ + r_-)r + (r_+ + r_-)^2 - r_+r_-| \Bigg). \end{aligned} \tag{10}$$

When $Q^2 = \frac{3}{8}(3M^4/2\pi|\Lambda|)^{1/3}$, $r_+ = r_- =: r_\mathrm{h}$,

$$r^* = \frac{1}{2\Lambda} \frac{1}{r - r_\mathrm{h}} - \frac{7}{6\sqrt{2}\Lambda} \frac{1}{r_\mathrm{h}} \arctan \frac{r + r_\mathrm{h}}{\sqrt{2} r_\mathrm{h}} + \frac{1}{3\Lambda} \frac{1}{r_\mathrm{h}} \log \frac{(r + r_\mathrm{h})^2 + 2r_\mathrm{h}^2}{(r - r_\mathrm{h})^2}. \tag{11}$$

In the retarded and advanced coordinate systems, (3) has the form

$$\mathrm{d}s^2 = \left( -\tfrac{1}{3}\Lambda r^2 - \frac{2M}{\pi r} + 4\frac{Q^2}{\pi r^2} \right) \mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r - r^2(\mathrm{d}\varphi^2 + \mathrm{d}\psi^2), \tag{12}$$

$$\mathrm{d}s^2 = \left( -\tfrac{1}{3}\Lambda r^2 - \frac{2M}{\pi r} + 4\frac{Q^2}{\pi r^2} \right) \mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r - r^2(\mathrm{d}\varphi^2 + \mathrm{d}\psi^2), \tag{13}$$

respectively. They are regular at $r_\pm$. In fact, the surface with $r_\pm$ is nothing but the static surface and the event and Cauchy horizon as will be seen from the Penrose diagram.

It can be shown that the radial null geodesic equation is

$$(\mathrm{d}r/\mathrm{d}\lambda)^2 = p_t^2, \tag{14}$$

where $\lambda$ is the affine parameter of the null geodesics and $p_t = g_{tt}\mathrm{d}t/\mathrm{d}\lambda$ is a constant of motion. For a given $p_t$, $\lambda(r=0) - \lambda(r) = \mp r/p_t$ is finite. After a straightforward calculation, one finds

$$C^{ab}{}_{cd}C^{cd}{}_{ab} = \frac{38}{\pi^2 r^6}(M - 4Q^2/r)^2, \tag{15}$$

where $C^{ab}{}_{cd}$ is the Weyl tensor. Therefore, all radial null geodesics are incomplete and $r = 0$ is the s.p. curvature singularity of the space–time. By the way, the radial time-like geodesics, like those in the Reissner–Nordström space–time, skirt the singularity and emerge in other domains.

We now discuss the "Penrose diagram" of the space–time. The topology of the space–time containing a single black hole is usually $S^2 \times \mathcal{M}^2$ and the Penrose diagrams were invented originally to describe the structures of the non-trivial two-dimensional manifold $\mathcal{M}^2$. In our case the topology is $S \times S \times \mathcal{M}^2$ and a causal diagram similar to the usual Penrose diagram describing the structures of the non-trivial manifold $\mathcal{M}^2$ should also be helpful.

Consider the conformal transformation, $\mathrm{d}s^2 = r^2\,\mathrm{d}\bar{s}^2$, and coordinate transformation, $l = 1/r$, then one obtains

$$\mathrm{d}\bar{s}^2 = \left( -\tfrac{1}{3}\Lambda - \frac{2M}{\pi}l^3 + 4\frac{Q^2}{\pi}l^4 \right)\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}l - (\mathrm{d}\varphi^2 + \mathrm{d}\psi^2), \tag{16}$$

$$\mathrm{d}\bar{s}^2 = \left( -\tfrac{1}{3}\Lambda - \frac{2M}{\pi}l^3 + 4\frac{Q^2}{\pi}l^4 \right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}l - (\mathrm{d}\varphi^2 + \mathrm{d}\psi^2), \tag{17}$$

which are regular at $l = 0$. When $Q^2 < \frac{3}{8}(3M^4/2\pi|\Lambda|)^{1/3}$, define new coordinates

$$\begin{aligned} U &= \arctan(\mp\beta^{-1}\exp\{\tfrac{1}{6}\Lambda(r_+ - r_-)[2 + (1+\alpha)^2](t - r^*)\}], \\ V &= \arctan(\pm\beta^{-1}\exp\{-\tfrac{1}{6}\Lambda(r_+ - r_-)[2 + (1+\alpha)^2](t + r^*)\}), \end{aligned} \tag{18}$$

for the regions $r_+ < r$ and $0 < r < r_-$

$$\begin{aligned} U &= \arctan(\pm\beta^{-1}\exp\{-\tfrac{1}{6}\Lambda(r_+ - r_-)[2 + (1+\alpha)^2](t + r^*)\}), \\ V &= \arctan(\pm\beta^{-1}\exp\{-\tfrac{1}{6}\Lambda(r_+ - r_-)[2 + (1+\alpha)^2](-t + r^*)\}), \end{aligned} \tag{19}$$

for the regions $r_- < r < r_+$, where

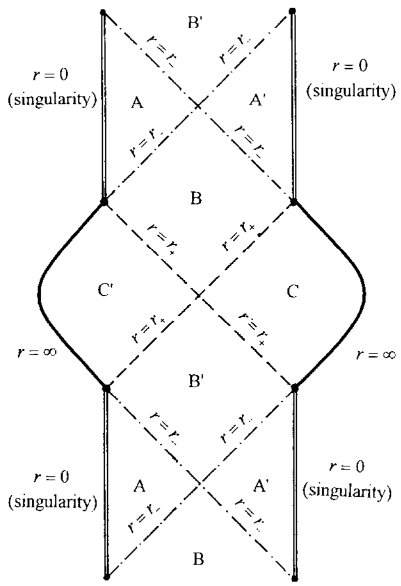
$$\begin{aligned} \beta &= r_+^{-1/2} r_-^{\alpha} (r_+^2 + r_+ r_- + r_-^2)^{-(1-\alpha)(1+\alpha)^3[(1+\alpha)^2+2]/2[4(1+\alpha)^4+2(1+\alpha^2)^2]} \\ &\times \exp\left( -\frac{1-\alpha}{2} \frac{(1+\alpha)^4 + 2(1+\alpha^2)(1+\alpha+\alpha^2)}{[(1+\alpha)^2 + 2\alpha^2][(1+\alpha)^2 + 2(1+\alpha^2)]^{1/2}} \right. \\ &\left. \times \arctan\frac{1+\alpha}{[(1+\alpha)^2 + 2(1+\alpha^2)]^{1/2}} \right) \end{aligned} \tag{20}$$

and $\alpha = r_-/r_+$. The upper sign in (18) and (19) corresponds to the regions A, B, C and the lower sign to the regions A′, B′, C′ in Fig. 1. When $Q^2 = \frac{3}{8}(3M^4/2\pi|\Lambda|)^{1/3}$, define new coordinates

$$U = \arctan[-\beta^{-1}\exp(-u)], \qquad V = \arctan[\beta^{-1}\exp(v)], \tag{21}$$

where

$$\beta = \exp\left( -\frac{1}{\Lambda}\frac{1}{r_\mathrm{h}}(\tfrac{1}{2} + \tfrac{7}{12}\sqrt{2}\arctan\tfrac{1}{2}\sqrt{2} - \tfrac{1}{3}\log 3) \right). \tag{22}$$



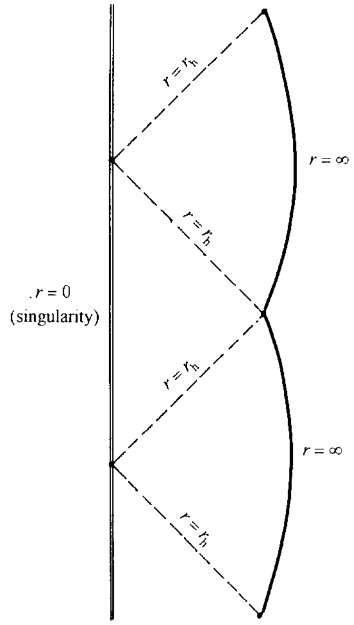




Fig. 1. The "Penrose diagram" for the static charged torus-like black hole with $r_+ = 2r_- = 2$.

Fig. 2. The "Penrose diagram" for the static extremely charged torus-like black hole with $r_h = r_+ = r_- = -1/\Lambda$.

Figs. 1 and 2 are the "Penrose diagrams" for the static charged torus-like black hole. Note that each point on the diagrams represents a torus rather than a two-sphere. This "Penrose diagram" is similar to that of the Reissner–Nordström space-time in many important respects: any signals which pass across $r = r_+$ (or $r = r_h$ for the $r_- = r_+$ case) will not be able to get back again to our universe; the singularities are time-like; the Killing vector $\partial/\partial t$ is time-like near the singularities; although null geodesics can reach the singularities (hence incomplete), no time-like geodesics can; the existence of quite a lot of non-geodetic time-like curves reaching the singularities does not mean that observers in a rocket-ship with enough fuel can reach there, since the divergence at the singularities of the norm of the time-like Killing field $\partial/\partial t$ implies, according to Ref. [8], that to reach the singularities the rocket-ship must have an infinite amount of fuel.

In conclusion, there exists a static black hole solution of the Einstein–Maxwell equation, whose event horizon has $S^1 \times S^1 \times \mathbb{R}$ topology. The existence of such a black hole does not violate the topology theorem since each surface of this space–time at constant $t$ and $r$ has a toroidal topology which is essentially different from that of the asymptotically flat space–times. In other words, the assumption of the theorem is not satisfied. Besides, one may construct from this torus-like black hole the cylindrically symmetric "black hole" by undoing the identification of $\psi = 0$ with $\psi = 2\pi$ [9], and construct the plane-symmetric "black hole" by further undoing identification of $\varphi = 0$ with $\varphi = 2\pi$. It might be of interest to study the stability, the quantum effects near the event horizon, and black hole thermodynamics of these black holes.

Finally, we would like to make some comments on the physical relevance of the torus-like black hole. First of all, the Universe might well be multiply-connected [10], in which the torus-like black hole might have been formed and survived. Secondly, one cannot rule out the possibility, at least at present, that some of the energy conditions are violated either by the existence of some bizarre matters or by taking into consideration the quantum effects of matter fields. Therefore, it is not impossible that a torus-like black hole has been connected in some strange way to an asymptotically flat space–time to form an asymptotically flat space–time containing a torus-like black hole. As the last comment, the tunnelling effects between different topologies should not be neglected when gravity is quantized.

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## References


[1] B. Carter, in: Black holes, eds. C. DeWitt and B.S. DeWitt (Gordon and Beach, New York, 1973) p. 57.
[2] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time (Cambridge Univ. Press, Cambridge, 1973).
[3] J.L. Friedman, K. Schleich and D.M. Witt, Phys. Rev. Lett. 71 (1993) 1486;
G. Galloway, Commun. Math. Phys. 151 (1993) 53;
P.T. Chruściel and R.M. Wald, On the topology of stationary black holes, e-print gr-qc/9410004.
[4] H. Geonner and J. Stachel, J. Math. Phys. 11 (1970) 3358.
[5] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, Exact solutions of Einstein's field equation (Cambridge Univ. Press, Cambridge, 1980).
[6] C. Kar, Phys. Z. 27 (1926) 208.
[7] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506;
J.D. Brown, J. Creighton and R.B. Mann, Temperature, energy and heat capacity of asymptotically anti-de Sitter black holes, preprint WATPHYS TH-94/02, CTMP/006/NCSU.
[8] S. Chacrabarti, R. Geroch and C.B. Liang, J. Math. Phys. 24 (1983) 597.
[9] C.G. Huang, Charged static cylindrical black hole, Acta Phys. Sin. (Oversea's edition), to be published.
[10] L.Z. Fang and H. Sato, Gen. Rel. Grav. 17 (1985) 1117.