Generalized Counting Rule for Hard Exclusive Processes

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We derive a generalized counting rule for hard exclusive processes involving parton orbital angular momentum and hadron helicity flip. We start with a systematic way to enumerate the Fock components of a hadronic light-cone wave function with \( n \) partons and orbital angular momentum projection \( l_z \). We show that the wave-function amplitude \( \psi_n(x_i, k_{i\perp}, l_i, l_z) \) has a leading behavior \( 1/(k_{1\perp}^2)^{[n+1]l_z+\min(n'+[l_z])}/2-1 \) when all parton transverse momenta are uniformly large, where \( n' \) and \( l_z' \) are the number of partons and orbital angular momentum projection, respectively, of an amplitude that mixes under renormalization. Besides the generalized counting rule, the result can be used as a constraint in modeling the hadronic light-cone wave functions.

\[
\psi_n(x_i, k_{i\perp}, l_i, l_z) \sim \frac{1}{(k_{1\perp}^2)^{[n+1]l_z+\min(n'+[l_z])}/2-1},
\]

in the limit that \( k_{1\perp} \sim k_{2\perp} \sim \ldots \sim k_{n-1\perp} \sim k_{\perp} \rightarrow \infty \), where \( n' \) and \( l_z' \) characterize the amplitude that mixes under scale evolution. The result explains the scaling behavior of the \( F_2(Q^2) \) form factor obtained recently in perturbative QCD [13], and helps to establish more general scaling properties of exclusive scattering amplitudes [14-18]. It also can be used as a constraint in building phenomenological wave functions of hadrons consistent with perturbative QCD.

Let us first introduce a systematic method to enumerating the light-cone Fock wave function of a hadron with helicity \( \Lambda \). Suppose a Fock component has \( n \) partons with creation operators \( a_i^\dagger \), \( a_i \), where the partons can either be gluons or quarks and the subscripts label the partons’ quantum numbers such as spin, flavor, color, momentum, etc. Assume all color, flavor (for quarks) indices have been coupled properly using Clebsch-Gordon coefficients. The longitudinal momentum fractions of the partons are \( x_i \) \( (i = 1, 2, \ldots, n) \), satisfying \( \sum_{i=1}^n x_i = 1 \), and the transverse momenta \( \vec{k}_{i\perp} \), satisfying \( \sum_{i=1}^n \vec{k}_{i\perp} = 0 \). We will eliminate \( \vec{k}_{\perp} \) in favor of the first \( n-1 \) transverse momenta. Assume the orbital angular momentum projections of the partons are \( l_{z1}, \ldots, l_{zn-1} \), respectively, and let \( l_z = \sum_{i=1}^{n-1} l_{zi} \), then

\[
l_z + \lambda = \Lambda,
\]

where \( \lambda = \sum_{i=1}^{n-1} \lambda_i \) is the total parton helicity. Without loss of generality, we assume \( l_z \geq 0 \); even then, \( l_{zi} \) can have both signs. Thus, a general term in the hadron wave function has the structure
where $k_{1\perp} = k_1^z \pm k_1^z$ and the $+$ sign applies when $l_{zj}$ is positive (negative), and $d[i] = dx_1 dx_2 l_{1\perp} / [\sqrt{2\xi (2\pi)^3}]$ with the overall constraint on $x_1$ and $k_{1\perp}$ implicit.

The above form can be further simplified as follows. Assume $l_{zj}$ is positive and $l_{zj}$ negative, and $l_{zj} > |l_{zj}|$, we have

$$(k_{1\perp}^+)^{l_{zj}} (k_{2\perp}^+)^{l_{zj}'} = (k_{1\perp}^+)^{l_{zj}'} (k_{2\perp}^+)^{l_{zj}}$$

where $a, b, c$ are polynomials in $k_{1\perp}, k_{2\perp},$ and $\vec{k}_{1\perp} \cdot \vec{k}_{2\perp}$. On the last line of the above equation we have used the identity $e^{a+b}e^{b} = e^{a+b}e^{b}$. If $l_{zj} + l_{zj}' \neq 0$, one can use $i e^{a+b}k_{1\perp}k_{2\perp}k_{1\perp}' = k_{1\perp} \cdot k_{2\perp}k_{1\perp}' - \vec{k}_{1\perp} \cdot \vec{k}_{1\perp}'$ to further reduce the second term in the bracket. Following the above procedure, we can eliminate all negative $l_{zj}$, a general $l > 0$ component in the wave function reads

$$\int \prod_{i=1}^{n} d[i](k_{1\perp}^i)^{l_{zj}} (k_{2\perp}^i)^{l_{zj}'} \cdots [k_{(n-1)\perp}^i]^{l_{z(n-1)}} \times \sum_{i<j}^{n} \int_{l_{zj}+l_{zj}'} \psi_n(x_i, k_{i\perp}, l_i) a_1^i a_2^i \cdots a_n^i |0\rangle,$$

where $\sum_{i} l_{zj} = l$ and $l_{zj} \geq 0$, and the sums over $i$ and $j$ are restricted to the $l_{zj} = 0$ partons.

Using the above procedure, it is easy to see that the proton state with three valence quarks has six independent scalar amplitudes $\psi_n^{(i \leftarrow 0)}$, $i = 1, \ldots, 6$ [11]. The wave-function amplitudes for three quarks plus one gluon will be presented in a separate publication.

The mass dimension of $\psi_n$ can be determined as follows: Assume the nucleon state is normalized relativistically $\langle P'|P \rangle = 2E(2\pi)^3 \delta^3(P' - P)$. $|P\rangle$ has mass dimension $-1$. Likewise, the parton creation operator $a_i^\dagger$ has mass dimension $-1$. Given these, the mass dimension of $\psi_n$ is $-(n + |l_{zj}| - 1)$. The mass dimension of $\psi_n(i\leftarrow j)$ on the other hand, is $-(n + |l_{zj}| + 1)$ which can be accounted for by the previous formula with an effective angular momentum projection $|l_{zj}| + 2$.

To find the asymptotic behavior of an amplitude $\psi_n(x_i, k_{i\perp}, l_i)$ in the limit that all transverse momenta are uniformly large, we consider the matrix element of a corresponding quark-gluon operator between the QCD vacuum and the hadron state

$$\langle 0 | \phi_{\mu_1}(\xi_1) \cdots \phi_{\mu_n}(\xi_n)|\Lambda\rangle,$$

where $\phi$ are parton fields such as the “good” ($+$) components of quark fields or $F^{+g}$ of gluon fields, and $\mu_i$ are Dirac and transverse coordinate indices when appropriate. All spacetime coordinates $\xi_i$ are at equal light-cone time, $\xi_i^+ = 0$. Fourier transforming with respect to all the spatial coordinates ($\xi_i^-, \xi_{i\perp}$), we find the matrix element in the momentum space, $\langle 0 | \phi_{\mu_1}(k_1) \cdots \phi_{\mu_n}(k_{n-1}) \times \phi_{\mu_n}(0)|\Lambda\rangle \equiv \psi_{\mu_1 \cdots \mu_n}(k_1, \ldots, k_{n-1})$, here we have just shown $n-1$ parton momenta because of the overall momentum conservation. The matrix element can be written as a sum of terms involving projection operator $\Gamma_{\mu_1 \cdots \mu_n}(k_{1\perp})$ multiplied by scalar amplitude $\psi_{n\mu}(x_i, k_{i\perp}, l_i)$:

$$\langle 0 | \phi_{\mu_1}(k_1) \cdots \phi_{\mu_n}(k_{n-1}) \phi_{\mu_n}(0)|\Lambda\rangle \equiv \psi_{\mu_1 \cdots \mu_n}(k_1, \ldots, k_{n-1}) = \sum A^\dagger_{\mu_1 \cdots \mu_n}(k_{1\perp}) \psi_n^{(\text{asy})}(x_i, k_{i\perp}, l_i),$$

where the projection operator $\Gamma^\dagger$ contains Dirac matrices and is a polynomial of order $|l_{zj}|$ in parton momenta. For example, the two-quark matrix element of the pion can be written as [9],

$$\langle 0 | \overline{\alpha}_{\mu}(0) \alpha_{\nu}(x, k_{\perp}) (\pi^+ (P) = (\gamma_5 \gamma^\nu \psi_n^{(\text{asy})}(x, k_{\perp}, l_z = 0) + (\gamma_5 \gamma^\nu \gamma^\sigma \psi_n^{(\text{asy})}(x, k_{\perp}, l_z = 1),$$

where the projection operators are shown manifestly. More examples for the proton matrix elements can be found in Ref. [11].

The matrix element of our interest is, in fact, a Bethe-Salpeter amplitude projected onto the light cone. One can write down formally a Bethe-Salpeter equation which includes mixing contributions from other light-cone matrix elements. In the limit of large transverse momentum $k_{1\perp}$, the Bethe-Salpeter kernels can be calculated in perturbative QCD because of asymptotic freedom. In the lowest order, the kernels consist of a minimal number of gluon and quark exchanges linking the active partons. For the lowest Fock components of the pion wave function, one gluon exchange is needed to get a large transverse momentum for both quarks [17]. As we shall see, asymptotic behavior of the wave-function amplitudes depends on just the mass dimension of the kernels.

Schematically, we have the following equation for the light-cone amplitudes,
where $H_{a_1 \ldots a_n \beta_1 \ldots \beta_d}$ are the Bethe-Salpeter kernels multiplied by the parton propagators. When the parton transverse momenta are uniformly large, the kernels can be approximated by a sum of perturbative diagrams. The leading contribution to the amplitudes on the left can be obtained by iterating the above equation, assuming the amplitudes under the integration sign contain no hard components. As such, the integrations over $q_{i\perp}$ can be cut off at a scale $\mu$ where $k_{\perp} \gg \mu \gg \Lambda_{\text{QCD}}$, and the $q_{i\perp}$ dependence in $H$ can be expanded in Taylor series. In order to produce a contribution to $\psi_n^{(A)}(x_i, k_{i\perp}, l_{zi})$, the hard kernels must contain the projection operator $\Gamma^A_{a_1 \ldots a_n}(k_1, \ldots, k_{n-1})$. Hence we write

$$H_{a_1 \ldots a_n \beta_1 \ldots \beta_d}(q_i, k_i) = \sum_{A,B} \Gamma^A_{a_1 \ldots a_n}(k_1) H_{AB}(x_i, k_{i\perp}, y_i) \times \Gamma^B_{\beta_1 \ldots \beta_d}(q_{i\perp}),$$

where $\Gamma^B_{\beta_1 \ldots \beta_d}(q_{i\perp})$ is again a projection operator and $H_{AB}(x_i, k_{i\perp}, y_i)$ are scalar functions of the transverse momenta $k_{i\perp}$ invariants. Substituting the above into Eq. (8) and integrating over $q_{i\perp}$, we have,

$$\psi_n^{(A)}(x_i, k_{i\perp}, l_{zi}) = \sum_{B,\beta} \int \prod_{i=1}^{n'} d[l_{i}] H_{AB}(x_i, k_{i\perp}, y_i) \Gamma^B_{\beta_1 \ldots \beta_n}(q_{i\perp}) \times \psi_{\beta_1 \ldots \beta_n}(y_i, q_i)$$

where the integrations over $q_{i\perp}$ are nonzero only when the angular momentum content of $\Gamma^B$ and $\Gamma^{A'}$ is the same. Now the large momenta $k_{i\perp}$ are entirely isolated in $H_{AB}$ which does not depend on any soft scale. The asymptotic behavior of $\psi_n^{(A)}(k_{i\perp})$ is determined by the mass dimension of $H_{AB}$, which can be obtained, in principle, by working through one of the simplest perturbative diagrams.

A much simpler way to proceed is to use light-cone power counting in which the longitudinal mass dimension, such as $P^+$, can be ignored because of the boost invariance of the above equation along the $z$ direction. We just need to focus on the transverse dimensions. Since the mass dimension of the amplitudes is $-(n + |l_{i\perp}| - 1)$, that of $\Gamma^B \Gamma^{A'}$ is $2|l_{i\perp}|$, and the integration measure $2^{n} |l_{i\perp}|$, a balance of the mass dimensions yields $H_{AB} = -(n - 1 + |l_{i\perp}| - (n' - 1 + |l_{i\perp}'|))$. Therefore, we arrive at the central result of our Letter Eq. (1) for the leading behavior of the wave-function amplitude, which is determined by a mixing amplitude with smallest $n' + |l_{i\perp}'|$. Since the wave function has mass dimension of $-(n + |l_{i\perp}| - 1)$, the coefficient of the asymptotic form must have a soft mass dimension $\Lambda_{\text{QCD}}^{\text{soft}(n' + |l_{i\perp}'| - 1)}$.

For the quark-antiquark amplitudes of the pion, the leading behavior is determined by self-mixing: $\psi^{(1)}_{\text{ud}}(x, k_{\perp}) \sim 1/k_{\perp}^2$ and $\psi^{(2)}_{\text{ud}}(x, k_{\perp}) \sim 1/(k_{\perp}^4)^2$. On the other hand, for the three-quark amplitudes of the proton [11], we have, $\psi^{(1)}_{\text{ud}}(x, k_{\perp}) \sim 1/(k_{\perp}^2)^3$, $\psi^{(2)}_{\text{ud}}(x, k_{\perp}) \sim 1/(k_{\perp}^4)^3$, $\psi^{(6)}_{\text{ud}}(x, k_{\perp}) \sim 1/(k_{\perp}^4)^4$. Here we recall that for $\psi^{(2)}_{\text{ud}}$, the effective angular momentum projection is $l_{\perp}^2 = 2$. Its leading behavior is determined by its mixing with $\psi^{(1)}_{\text{ud}}$.

The above method of enumerating light-cone wave functions with arbitrary orbital angular momentum projection and deriving their asymptotic behavior at large transverse momenta is applicable to any renormalizable quantum field theory. In the context of QCD, one of the important consequences is the power counting rule for the hadronic exclusive processes. While the traditional counting rule assumes that partons have zero orbital motion in hadrons [14,15], a generalized counting rule can be derived including nonzero orbital angular momenta. Let us consider a few examples here and leave a more detailed discussion to a longer publication.

The first important application is the Pauli form factor $F_2$ of the nucleon which involves hadron helicity flip. Since the quark mass effects are negligible, the helicity flip can be achieved through quark orbital angular momentum. Using the expression derived for $F_2(Q^2)$ in Ref. [11] and the asymptotic behavior of $\psi^{(1)}_{\text{ud}} \sim 1/k_{\perp}^2$ and $\psi^{(3,4)}_{\text{ud}} \sim 1/k_{\perp}^4$, we easily derive the result found in Ref. [13]:

$$F_2(Q^2) \sim 1/(Q^2)^3,$$

in asymptotic limit. The explicit perturbative QCD calculation finds an additional double logarithm in $Q^2$ dependence, and predicts that $Q^2/\log^2(Q^2/\Lambda^2)F_2/F_1$ scales as a constant at large $Q^2$ [13]. This latter scaling works better than expected with the recent JLab data [13,19,20]. We point out that the proton amplitudes $\psi^{(3,4)}_{\text{ud}}$ obtained from Melosh rotation are suppressed by only one power of $k_{\perp}$ relative to $\psi^{(1)}_{\text{ud}}$, and hence are inconsistent with perturbative QCD in the large $k_{\perp}$ limit [7,21–23].
A quick way to find a generalized counting rule for hard exclusive processes [14,15] is to count the soft mass dimensions in scattering amplitudes; the scaling in hard kinematic variables is then determined by dimensional balance. For example, the wave-function amplitude \( \psi_n(x_i, k_{i\perp}, l_{ij}) \) when used in a factorization formula contains a soft-scale factor \( \Lambda_{QCD}^{-n+|l_{ij}|-1} \). Therefore a scattering amplitude involving \( H = 1, \ldots, N \) hadrons with the light-cone amplitudes \( \psi_n(x_i, k_{i\perp}, l_{ij}) \) contains a soft mass factor \( \Lambda_{QCD}^{-n+|l_{ij}|-1} \). In the hadronic process \( A + B \rightarrow C + D + \cdots \), the fixed-angle scattering cross section calculated using the amplitudes \( \psi_n(x_i, k_{i\perp}, l_{ij}) \) goes like

\[
\Delta \sigma \sim s^{-1} \sum (n_H + |l_{ij}|-1) , \tag{12}
\]

where \( H \) sums over all hadrons involved and \( \Delta \sigma \) contains only angular variables. For \( l_{ij} = 0 \) and minimal \( n \), this is just the traditional counting rule of [14,15]. The generalized counting rule here applies to any hard process proceeded through any wave-function amplitudes including hadron helicity flip. It reproduces the result of Chernyak and Zhitnitsky for form factors where parton orbital angular momentum was first considered [16].

As a second application, we consider \( pp \) elastic scattering. Three helicity conservation amplitudes are known to go like \( M(+\rightarrow +\rightarrow +) \sim M(+\rightarrow -\rightarrow -) \sim M(-\rightarrow +\rightarrow -) \sim 1/s^4 \) [18]. Our counting rule provides the scaling behavior of the helicity flipping amplitudes \( M(+\rightarrow +\rightarrow -) \sim 1/s^{9/2} \) and \( M(-\rightarrow +\rightarrow -) \sim 1/s^5 \). The interference between different helicity amplitudes offers a new mechanism to explain the experimental observed oscillation in the scaling cross sections and the spin correlations [24].

Our final example is the fixed-angle photopion production, \( \gamma p \rightarrow \pi^+ n \), for which the hadron helicity conservation amplitudes scale \( M(\gamma p_1 \rightarrow \pi^+ n_1) \sim 1/s^{5/2} \). Our rule predicts that the hadron helicity-flip amplitude \( M(\gamma p_1 \rightarrow \pi^+ n_1) \sim 1/s^3 \). New experiments at JLab with polarizations allow a separation of different helicity amplitudes and thus a test of the generalized counting rule.

We end the Letter with a few cautionary notes. First, we have ignored the Lapshof type of contributions in hadron-hadron scattering [25]. Second, in an actual calculation of a scattering amplitude, there are integrations over partons’ light-cone fractions \( x_i \). These integrations may be divergent at the end points \( x_i = 0, 1 \) depending upon the choices of the light-cone wave functions. The QCD factorization and the naive power counting break down there [26,27]. Finally, the light-cone wave functions defined in the light-cone gauge have singularities [28]. When regularized, Sudakov type of form factors appear which lead to the dependence of the light-cone wave functions on \( P^+ \) [29]. The \( k_{\perp} \) counting breaks down in the region where the Sudakov form factors are important.

However, in certain cases the end point singularities are regulated by the Sudakov effects, and the last two adverse factors cancel [30], leaving the naive counting rule intact. It is not clear, however, that this happens in general.

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