On classifying the divisor involutions in Calabi-Yau threefolds

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ABSTRACT: In order to support the odd moduli in models of (type IIB) string compactification, we classify the Calabi-Yau threefolds with $h^{1,1} \leq 4$ which exhibit pairs of identical divisors, with different line-bundle charges, mapping to each other under possible divisor exchange involutions. For this purpose, the divisors of interest are identified as completely rigid surface, Wilson surface, $K3$ surface and some other deformation surfaces. Subsequently, various possible exchange involutions are examined under the symmetry of Stanley-Reisner Ideal. In addition, we search for the Calabi-Yau theefolds which contain a divisor with several disjoint components. Under certain reflection involution, such spaces also have nontrivial odd components in (1,1)-cohomology class. String compactifications on such Calabi-Yau orientifolds with non-zero $h^{1,1}_-(CY_3/\sigma)$ could be promising for concrete model building in both particle physics and cosmology. In the spirit of using such Calabi-Yau orientifolds in the context of LARGE volume scenario, we also present some concrete examples of (strong/weak) swiss-cheese type volume form.

Contents

1 Introduction

In order to describe the low energy physics in the real four dimensional world, string theory has to be compactified on a six dimensional manifold. To obtain supersymmetry in four dimensions, one requires Kählerity and the Ricci flatness of such manifolds which result in a Calabi-Yau manifold. For a realistic string model, especially in order to get a global model, one of the most challenging requirements is to stabilize the moduli associated with the Calabi-Yau compactifications. In the context of type IIB string compactifications (for a review, see [1, 2]), there are two classes of mechanism for moduli stabilization; namely the KKLT [3] and the LARGE volume scenario [4]. In both of these mechanisms, complex structure moduli and axio-dilaton are stabilized at tree level by Gukov-Vafa-Witten(GVW) flux contributions to the superpotential [5] (see related work [6, 7]) while the Kähler moduli are stabilized with the inclusion of non-perturbative superpotential corrections coming from $E3$-instanton or gaugino condensation [8]. In LARGE volume scenarios, a perturbative (α'^3) correction to the Kähler potential [9] is balanced against the exponentially suppressed non-perturbative superpotential corrections, and subsequently leads to exponentially large stabilized value for the overall volume of the Calabi-Yau.

Nowadays, people are making great effort to combine global issues like moduli stabilization and tadpole/anomaly cancellation with the local model building issues such as

getting the GUT or MSSM-like spectrum. There are several technical aspects which should be taken care of. One of the crucial issues is the conflict between chirality in visible sector and survival of the non-perturbative superpotential contribution [10]. That is, when one considers a visible sector in a compactification scheme, there are charged instanton zero modes coming from intersection between E3-brane and D7-brane with world-volume flux, which will prevent a class of instantons from participating in moduli stabilization. Several efforts have been made to avoid such a problem [11–17]. One way is, not to support the visible sector on the divisor which gives rise to the non-perturbative superpotential contribution. Along these lines, models supporting visible sector with D-branes at singularities have been of interest [15–17]. This singularity arises from D-flatness condition which forces one or more four-cycles shrink to zero size if there are no visible sector singlets which can get a non-vanishing VEV to compensate the Fayet-Ilopoulos(FI) term in the D-term potential. In this approach, one needs to embed such singularities in a compact Calabi-Yau threefold which has to contain non-zero odd components in cohomology class $H^{1,1}_-(CY_3/\sigma)$ under some holomorphic involution σ. Another way to alleviate the chirality issue has been argued in a generalized setup which includes nontrivial involutively odd two-cycles to support instanton flux [13] which again requires non-zero $H^{1,1}_-(CY_3/\sigma)$. More specifically, equipped with the fluxed-insatnton contribution, one includes the involutively odd-moduli (b^a, c^a) which arise from the NS-NS field B_2 and R-R field C_2 in type IIB orientifolds to correct the E3-brane superpotential and then remove the extra charged zero modes. These b^a and c^a moduli appear when there is a nontrivial splitting of cohomology $H^{1,1}(CY_3)$ under holomorphic involution σ, and are counted by $h^{1,1}_-(CY_3/\sigma)$. These odd moduli will appear in the Kähler potential as well as in the superpotential, and hence can help in moduli stabilization. In [18], the moduli stabilization of these odd moduli is studied in detail in the context of LARGE volume scenario. So in either approach, finding Calabi-Yau threefolds with $h^{1,1}_-(CY_3/\sigma) \neq 0$ is a crucial ingredient, and is very useful in realizing some particular extensions of LARGE volume scenario with the inclusion of odd moduli, which could be promising for both particle physics and cosmology. This is the main purpose of this paper.

In the context of toric geometry, the simplest way to have nontrivial (1,1)-cohomology in the involutively odd sector is to exchange pairs of "Nontrivial Identical Divisor(NID)" by requiring $\sigma : x_i \leftrightarrow x_j$, where x_i is the homogeneous coordinates and the divisor $D_i \equiv \{x_i = 0\}$ is dual to the two-form $\hat{D}_i$. The holomorphic involution $\sigma : x_i \leftrightarrow x_j$, exchanging two homogeneous coordinates of the defining toric Calabi-Yau threefold, implies a pair of new basis divisors $D_\pm \equiv D_i \pm D_j \in H^{2,2}_\pm(CY_3)$ respectively. The two-form $\hat{D}_- \in H^{1,1}_-(CY_3)$ dual to the divisor D_- will support the odd moduli b^a and c^a, $a = 1, \ldots h^{1,1}_-$.[1]

In such cases, the volume of orietnifold odd four-cycles τ_- corresponding to the odd divisor D_- will shrink to zero and not appear in the Calabi-Yau volume form which also implies the splitting $h^{1,1}_+(CY_3/\sigma) + h^{1,1}_-(CY_3/\sigma)$. Here, the pairs of NIDs means two divisors with different line-bundle charges (also known as Gauged Linear Sigma Model(GLSM) charges) intersecting with the Calabi-Yau hypersurface to the same topological surface with the same Hodge number. The requirement of "Nontrivial Identical" comes from the fact

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[1]In the following section, we will drop the 'hat' for the dual two-form $\hat{D}_i$.

that the two divisors with the same GLSM charge are not distinguishable due to the same equivalence relations in the toric data. As a consequence, exchanging two divisors with the same GLSM charge will not affect the hypersurface polynomial and so will not contribute to $h^{1,1}_-(CY_3/\sigma)$. Some examples with involutions in elliptic fibered Calabi-Yau threefolds are studied in [19, 20]. Exchange of del-Pezzo divisors in the Calabi-Yau orientifold compactifications with or without K3 fiberation have been studied in [21, 22]. Using exchange involutions, a procedure has been developed in [23] for constructing F-theory fourfold uplifts of concrete models of perturbative type IIB (with O3/O7) Calabi-Yau orientifolds.

In the standard approach of moduli stabilization in LARGE volume type IIB orientifolds, many models have been constructed with reflection involutions $\sigma : x_i \to -x_i$. Usually such a reflection acts trivially on the homology of the Calabi-Yau space and results in $h^{1,1}_-(CY_3/\sigma) = 0$, therefore does not support odd moduli in the spectrum. However, there are exceptions if the Calabi-Yau space is *non-toric* in which reflection involutions can result in $h^{1,1}_-(CY_3/\sigma) \neq 0$. These *non-toric* spaces contain divisor with disconnected pieces like $\mathbb{P}^n \sqcup \cdots \sqcup \mathbb{P}^n$ or $dP_n \sqcup \cdots \sqcup dP_n$. We will show that under some reflection, these pieces exchange to each other and split to $h^{0,0}_+(D) + h^{0,0}_-(D)$, which will also contribute to the equivariant cohomology $h^{1,1}_-(CY_3/\sigma)$.

Motivated by the possibility of promising utilities of odd moduli in the type IIB extended LARGE volume scenario, our main goal in this article is to present a systematic classification of the possible exchangeable involutions for toric Calabi-Yau threefolds, and reflection for *non-toric* Calabi-Yau spaces with $h^{1,1}(CY_3) \leq 4$. Here, it is important to mention that a complete classification of 'all' genuine exchange involutions is highly nontrivial, and our simplistic approach captures only a subset of such an involutions. For classification purpose, we will scan through the list of Calabi-Yau hypersurfaces of toric fourfolds encoded as four-dimensional polytopes in Kreuzer-Skarke list [24] for $h^{1,1} \leq 4$. We will proceed with maximally triangulating all these reflexive polytopes and subsequently, get the triple intersection number along with the Kähler cone for each of these triangulations. Using these toric data, we will calculate all the Hodge numbers for the coordinate divisors on the hypersurface to see if they contain pairs of NIDs or divisors with disjoint pieces. For all the spaces (both simplicial and non-simplicial) containing NIDs, we will check whether there exists divisor exchange involution which is consistent with the symmetry of the Stanley-Reisner Ideal (SR-Ideal). Then we will also calculate the volume form to see such splitting $h^{1,1}_+ + h^{1,1}_-$ under such involution. For *non-toric* spaces, the volume form contains less information due to the *non-toric* property. We will directly calculate the representation of cohomology $H^{1,1}(CY_3)$ to see under which reflection it will split to odd part and the possibility to contribute as odd modulus.

The article is organized as follows: in section 2, we present the type of divisors which we consider in the later classification and their physical background. In section 3, we perform a classification of Calabi Yau spaces with $h^{1,1}(CY_3) \leq 4$ for the possible exchange involutions (and for possible reflection involution for the *non-toric* spaces) which could result in $h^{1,1}_-(CY_3/\sigma) \neq 0$. In section 4, we present the expression of volume forms for all the classes of exchangeable involutions discussed in section 3. Some of the volume forms presented are of swiss-cheese type. In section 5, we discuss the volume form for the *non-toric*

Calabi-Yau spaces and calculate the representation of $H^{1,1}(CY/\sigma)$ explicitly. In the appendices (A)–(C), We give sketch of computational tools utilized, and tabulate all the Calabi-Yau threefolds with explicit exchange (for $h^{1,1}(CY_3) \leq 3$) and reflection (for $h^{1,1}(CY_3) \leq 4$) involutions which may result in $h^{1,1}_-(CY_3/\sigma) \neq 0$. The list for all possible exchange involutions and the other relevant topological data for spaces with $h^{1,1} = 4$ are collected in the external file "Classification of Calabi-Yau Threefolds with Divisor Exchange Involution".

2 The geometry of divisors

In order to support the odd moduli in a string compactification, we are looking for certain Calabi-Yau spaces with $h^{1,1}_-(CY_3/\sigma) \neq 0$ which can either exchange pairs of "Nontrivial Identical Divisor(NID)" or contain a divisor with several disjoint components. The internal geometries for these divisors play an extremely important role in the process of compactification and subsequent moduli stabilization mechanisms. The most common divisors which can be investigated for exchange involution are completely rigid divisor, "Wilson" divisor and deformation divisor. We will classify the divisor exchange involution by these divisors in the next section.

- Completely rigid divisor.

  Completely rigid divisor means a divisor with Hodge numbers given as $h^\bullet(D) \equiv \{h^{0,0}, h^{0,1}, h^{0,2}, h^{1,1}\} = \{1, 0, 0, h^{1,1}\}$ such that $h^{1,1} \neq 0$. Depending on whether $h^{1,1}(D)$ is larger than 9 or not, these divisors are further distinguished as del-Pezzo surfaces $\{ \mathbb{P}^2 \equiv dP_0 , dP_n, \text{ with } n = 1, \ldots, 8 \}$ and "rigid but not del-Pezzo" surfaces $\{dP_n, \text{ with } n > 9\}$. The del-Pezzo divisors are obtained after blowing up (a set of) generic points in a $\mathbb{P}^2$. Such divisors can be either shrinkable or non-shrinkable depending on their intersections with the other four-cycles. For shrinkable del-Pezzo divisors, one can always find a basis for a given Calabi Yau threefold such that the only non vanishing intersection of the del-Pezzo divisor is the self-intersection.[2] This property can be observed in the volume form.

  On the physical side, having Euler character equal to one is the sufficient condition to have exactly two fermionic zero modes and therefore a nonzero contribution to the superpotential [8]. The completely rigid divisors satisfy this condition trivially. In particular, the shrinkable del-Pezzo surfaces are very important in string phenomenology. On the particle pheno side, it is argued that such shrinkable divisors are needed to support the GUT 7-branes in order to decouple from gravity. On the cosmology side, it is also helpful to realize the swiss-cheese structure in LARGE volume scenario [4, 25]. More recently, in the context of type IIB orientifold with $h^{1,1}_-(CY_3) \neq 0$, the identical shrinkable del-Pezzo surfaces result in a singularity, which is crucial for global model building on branes at del-Pezzo singularity [15–17].

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[2]This is also the reason that sometimes shrinkable del-Pezzo divisors are called as 'diagonal del-Pezzo' while non-shrinkable ones are called as 'non-diagonal del-Pezzo' [22].

- "Wilson" divisor.

  "Wilson" divisor means a divisor with $h^{1,0}(D) \neq 0$. Here we foucs on the following "Wilson" surface of which the Hodge Diamond is $h^{\bullet}(D) = \{1, h^{1,0}, 0, h^{1,1}\}$ with $h^{1,0}(D), h^{1,1}(D) \neq 0$. In our scanning, we will specify a particular "Wilson" surface as "Exact-Wilson" divisor like $h^{\bullet}(D) = \{1, 1, 0, h^{1,1}\}$ with $h^{1,1}(D) \neq 0$. The physical significance of such divisor comes from the so-called poly-instanton effect.[3]

  In moduli stabilization, as one has to consider a sum over all possible instanton contributions, it is a highly nontrivial task to ensure that the considered corrections are the only possible ones. In the context of type IIB orientifolds, it has been shown that in the presence of an Exact-Wilson divisor with $h_{+}^{1,0}(D) = 1$, one has the right zero mode structure for an Euclidean D3-brane wrapping on it to generate poly-instanton effect in the superpotential [26, 29]. Such new non-perturbative effects will result in a new class of Kähler moduli inflation called Poly-instanton inflation which treats the "exact-Wilson" divisor volume (or together with its axion part) as inflaton.

- Deformation divisor.

  Deformation divisor means a divisor with $h^{2,0}(D) \neq 0$. It has been proposed that turning on world-volume fluxes, it is possible to lift these (extra) deformation zero modes [32] while leaving the poly-instanton zero modes encoded in $h_{+}^{1,0}(D)$ unaffected. Since these fluxes can rigidify some deformation divisors, such circumstances facilitate the moduli stabilization process by introducing more terms for superpotential contributions. Such a mechanism has been utilized to build an explicit de Sitter in [33]. Here we focus on the following three kinds of deformation divisors.

  K3 surface: the Hodge diamond of a K3 divisor is $h^{\bullet}(D) = \{1, 0, 1, 20\}$. K3 surface present in a Calabi-Yau may or may not be fibered. A K3-fibred Calabi-Yau compactification is very helpful to obtain an anisotropic shape of the Calabi-Yau. This leads to some LVS models with effectively two large extra dimensions of micron size [22, 34]. The property of spaces which contain both $K3$ and Wilson surface are also studied in [29, 35].

  Special deformation surface: in our scanning, there are two kinds of deformation divisors which appear very frequently. One has an extra $h^{1,1}$ deformation for the K3 surface as $h^{\bullet}(D) = \{1, 0, 1, 21\}$ which will be labeled as $SD1$. The other one has a Hodge diamond of $h^{\bullet}(D) = \{1, 0, 2, 30\}$ which will be labeled as $SD2$. Both of these appear to have similar property in the volume form as K3 surface.

Now we turn to the *non-toric* spaces. As we described in the introduction, these spaces contain a divisor which consists of some disconnected pieces like $\mathbb{P}^n \sqcup \cdots \sqcup \mathbb{P}^n$ or $dP_n \sqcup \cdots \sqcup dP_n$. Under some reflection $\sigma : x_i \leftrightarrow -x_i$, these individual components D' and $D'' \ldots$ exchange to each other, and the combination of these will split to $h_{+}^{0,0}(D) + h_{-}^{0,0}(D)$. The rigid disjoint pieces imply that such splitting will also contribute to the equivariant

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[3]Poly-instanton means the correction of an Euclidean D-brane instanton action by other D-brane instantons [26–31].

cohomology $h^{1,1}_-(CY_3/\sigma)$ on the hypersurface. In these spaces, the number of Kähler cone generators and the number of toric equivalence relations are always smaller than $h^{1,1}(CY_3)$, and in the context of Type IIB orientifold one Kähler deformation is *non-toric*.

3 The classification of Calabi-Yau spaces

In this section, we classify the Calabi Yau threefolds for suitable exchange involutions. For this we follow a two-step approach. First, we search for all divisors which have identical hodge-diamond and different line bundle charges. This is what we call "Nontrivial Identical Divisor (NID)". In the second step, we combine all possible divisor exchange involutions and examine whether each of these involutions is consistent with the symmetry of SR-Ideal.

The reason for requiring the SR-Ideal to be invariant under divisor exchange involution is the fact that different Calabi-Yau hypersurfaces coming from different triangulations are encoded in different SR-Ideals. For requiring the exchange involution not to affect the triangulation, we demand the same to be a symmetry of the SR-Ideal for consistency.[4] Since this SR-Ideal symmetry is imposed on the ambient space and the hypersurface is invariant under involution, it will ensure that this symmetry is also preserved at the level of intersection form and Kähler cone condition on the hypersurface. More specifically, let us consider a simple example with $h^{1,1}(CY_3) = 3$, and suppose that the intersection form of the Calabi Yau hypersurface is written in a basis of divisors $\{D_i, D_j, D_k\}$ as $I_3 \equiv I_3(D_i, D_j, D_k)$ and the exchange involution is given as $\sigma : x_i \leftrightarrow x_j$. Then, in order to be a genuine involution, σ has to ensure that under the new basis $\{D_+, D_-, D_k\}$, where $D_\pm = D_i \pm D_j$, the intersection numbers with odd number of odd-indices ($\kappa_{kk-}, \kappa_{++-}, \kappa_{k+-}, \kappa_{---}$) should vanish for all k which can constitute an orientifold invariant basis. One can show that the Kähler cone condition will be consistently satisfied after orientifold involution. It is also important to note that since the divisors are given by the vanishing locus of some polynomials, SR-Ideal symmetry also ensures that we can exchange two deformation surfaces in a consistent way.

Let us illustrate this in a concrete example. Consider a Calabi Yau threefold defined by the following toric data

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
8	4	1	1	0	0	0	2
0	0	-1	-1	0	0	2	0
8	4	1	1	1	1	0	0
	$K3$	$K3$	$K3$	$K3$	W	W	

The Stanley-Reisner ideal reads

$$\text{SR} = \{x_2 x_3, x_4 x_5, x_1 x_6 x_7\}$$

Computing the Hodge number, one finds that the divisors D_2, D_3, D_4 and D_5 are K3, D_6, D_7 are Wilson line bundle. So the possible nontrivial involutions are $\{\sigma_1 : x_2 \leftrightarrow$

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[4]The condition of exchange involution being a symmetry of SR-Ideal might be stronger than the actual need, however we impose this to be on safe side.

$x_4, x_3 \leftrightarrow x_5\}$, $\{\sigma_2 : x_6 \leftrightarrow x_7\}$, $\{\sigma_3 : x_2 \leftrightarrow x_4, x_3 \leftrightarrow x_5, x_6 \leftrightarrow x_7\}$. One can see that all the three involutions are consistent with the symmetry of SR-Ideal. Expanding the Kähler form as $J = r^i[K_i]$, the Kähler cone is given simply by $r^i > 0$. One can show that the Kähler cone is generated by the divisors with following GLSM charges:

$$\{\{0, 0, 1\}, \{1, -1, 1\}, \{2, 0, 2\}\}.$$

We first analyze the involution $\{\sigma_1 : x_2 \leftrightarrow x_4, x_3 \leftrightarrow x_5\}$. Under the involution σ_1 we can choose one orientifold invariant basis to expand the Kähler form $J = t_i D_i$, which are $\{D_1, D_2, D_4\}$ (the same for D_3, D_5). For the Kähler parameters t_i the Kähler cone condition translates into

$$t_4 > 0, \, t_2 > 0, \, t_1 > 0. \tag{3.1}$$

Under the orientifold involution σ_1, we define $D_\pm = D_2 \pm D_4$ and the corresponding Kähler parameter $t_\pm$. Since the Kähler form is even under involution, i.e. $\sigma^*(J) = J$ in type IIB string framework and therefore belong to $H_+^{1,1}(CY_3)$, this implies that $J = t_1 D_1 + t_+ D_+$ and one should identify $t_2 = t_4$ to make $t_- = 0$. This identification is consistent with Kähler cone condition eq. (3.1). The intersection form under this basis reads

$$\begin{aligned} I_3 &= 32D_1^3 + 8D_1^2 D_2 + 8D_1^2 D_4 + 2D_1 D_2 D_4 \\ &= 32D_1^3 + 16D_1^2 D_+ + 4D_1 D_+^2 - 4D_1 D_-^2, \end{aligned}$$

from which the intersection numbers κ_{++-} and κ_{---} indeed vanish. All the results are consistent with the symmetry of the SR-Ideal.

Scanning set-up. In the following scanning, we will specify the Calabi-Yau spaces with $h^{1,1} \leq 4$ which are *non-toric* spaces or contain nontrivial divisor exchange involutions. We restrict to the toric ambient four-fold X_4 using the Kreuzer-Skarke list [24] of reflexive lattice polytopes and only consider the exchange of NID pairs as described in section 2. The Kreuzer-Skarke list contains 36, 244 and 1197 polytopes whose resulting threefolds are considered to have $h^{1,1}(CY_3) = 2$, 3 and 4 respectively. We consider all the maximal triangulations of such polytopes which result in 39, 342 and 2587 Calabi-Yau spaces.

On collecting these data, two things should be mentioned at the very outset. First, in triangulation, we don't take account the points in the dual-polytope which are interior to facets. This will not course problem since we are just interested in the Hodge number of the coordinate divisor on the Calabi-Yau hypersurface and these interior coordinate divisors don't intersect with the Calabi-Yau. Second, not all the maximal triangulations will correspond to distinguishable Calabi-Yau surface since there may exist flop-transitions of the ambient spaces which don't affect the hypersurface. Here, we don't distinguish them since it happens very rarely for small value of $h^{1,1}(CY_3)$.

The calculation is done by means of a scanning tool [36] for toric analysis (for a brief introduction see appendix A) which combines some properties of cohomCalg [37, 38],[5] PALP [39, 40] (with SINGULAR [41] extension) and the toric variety package of SAGE [42].

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[5]High-performance line bundle cohomology computation based on [37].

$h^{1,1}$ in Kreuzer-Skarke list	$\sharp$ of Polytopes	$\sharp$ of resulting CY_3 (Max.Tri)	$\sharp$ of CY_3 w. NIDs	$\sharp$ of constrained CY_3 w.NIDs
2	36	39	$2^{(7)}$	2
3	244	342	104	45
4	1197	2587	1419	396

Table 1. Number of CY_3 spaces with NIDs before and after imposing Stanley-Reisner Ideal symmetry. The last column does not contain *non-toric* spaces.

We collect the resolved vertex data in the reflexive dual-polytope without interior points of facets from PALP. Then fully triangulate them in SAGE and pick out the maximal triangulations. Using these toric ambient data we calculate the Hodge number of the coordinate divisor on the hypersurface by Cohomcalc [37, 38], Kähler cone generators from SAGE [42] and the triple intersection number from SINGULAR [41]. Based on the cohomology data of all the coordinate divisors we get, we can figure out which spaces contain NID and which one is *non-toric*. For all the spaces (both simplicial and non-simplicial) containing NIDs, we will check whether the exchange involution is allowed by the symmetry of SR-Ideal, and examine the consistency of orientifold invariant Kähler cone condition.

Scanning result. Now, we present the result of scanning for Calabi-Yau threefolds which permit exchange involution and reflection. Before imposing SR-Ideal symmetry, out of the 39, 342 and 2587 Calabi-Yau hypersurfaces, there are correspondingly 2, 104 and 1419 spaces which exhibit NIDs. From these NIDs appearing in the Calabi-Yau spaces, we can construct all possible divisor exchange involutions up to four pairs of divisor exchange, i.e. involve at most eight coordinate divisors which is sufficient for describing toric Calabi-Yau threefolds with $h^{1,1} \leq 4$. After requiring SR-Ideal symmetry of these involutions, the respective number of Calabi-Yau threefolds reduce to 2, 45 and 396[6] (see table 1). From these data, we can see the constraint of SR-Ideal symmetry dramatically decreases the number of suitable spaces which permit possible exchange involutions. The number of *non-toric* spaces is not affected by these constraints. There are 0, 1 and 16 spaces for $h^{1,1} = 2, 3$ and 4 respectively.

The classification of exchangeable involutions after the SR-Ideal symmetry constrains are classified in table 2. In this classification, we only consider the NID which corresponds to the surfaces discussed in section 2, i.e. completely rigid surface, Wilson surface and some deformation surfaces. Among these Calabi-Yau spaces, there are some which can also exchange several pairs of NIDs simultaneously. This is summarized in table 3. The relevant toric data along with the possible exchange involutions for spaces with $h^{1,1}(CY_3) = 2, 3$ are summarized in table 5–6 of the appendix B. For the *non-toric* spaces, we find that all the disconnected pieces appearing in the particular divisor are rigid surface. The counting of such spaces is presented in table 4 and the possible reflection under which $h^{1,1}_-(CY_3/\sigma) \neq 0$ is summarised in table 7 of the appendix C.

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[6]For these spaces, we didn't include *non-toric* spaces.

[7]In fact, these two spaces are identical.

$h^{1,1}$	Classification of constrained CY_3 with NIDs						
	Total	del-Pezzo dP_n, $n \leq 8$	dP_n, $n > 8$	Wilson (Exact)	K3	SD1	SD2
2	2	0	0	0	0	0	2
3	45	7	6	4(0)	35	2	2
4	396	151	232	31(3)	170	26	38

Table 2. Classification of constrained CY_3 with NIDs.

$h^{1,1}$	Classification of constrained CY_3 with several pairs of NIDs			
	Total	del-Pezzo & K3	del-Pezzo & Wilson(Exact)	K3 & Wilson(Exact)	del-Pezzo, K3 & Wilson(Exact)
2	2	0	0	0	0
3	45	0	1(0)	3(0)	0
4	396	2	21(0)	6(2)	0

Table 3. Classification of constrained CY_3 with several pairs of NIDs.

	$h^{1,1} = 2$	$h^{1,1} = 3$	$h^{1,1} = 4$
♯ of non-toric spaces	0	1	16

Table 4. Number of *non-toric* Calabi-Yau spaces.

The list of Calabi-Yau spaces with divisor exchange involution. For $h^{1,1}(CY_3) = 4$ we collect the result in the external file "Classification of Calabi-Yau threefolds with divisor exchange involution" together with $h^{1,1}(CY_3) = 3$ cases. Here, we provide the relevant assistance for understanding how these data are presented.

For convenience we collect $h^{1,1}(CY_3) = 4$ data in four groups in terms of the number of vertex in the reflexive dual-polytopes, i.e. from 5 to 8. In each of these groups we first present the toric data of CY_3 which permit nontrivial divisor exchange involutions. Then, we show the classification of these data as the discussion in table 2 and table 3.

Now, we give one example to illustrate how to read these data. This example is in the first place of the list $h^{1,1}(CY_3) = 4$ with 5 dual-vertex polytopes. Naively, it contains 2 basis of inequivalent involutions $\sigma_1 : \{x_4 \leftrightarrow x_5\}$, $\sigma_2 : \{x_7 \leftrightarrow x_8\}$. The first involution exchanges an identical divisor dP_7 and the second involution exchanges identical divisors of Wilson type. However, after checking the symmetry of SR-Ideal, only the combination of these two involutions $\sigma : \{x_4 \leftrightarrow x_5, x_7 \leftrightarrow x_8\}$ are relevant. In the list, we present these information as following:

- Toric data of relevant CY_3.

$$\Big\{ \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \{\{x_3, x_4\}, \{x_3, x_5\}, \{x_3, x_6\}, \{x_4, x_7\},$$
$$\{x_5, x_8\}, \{x_4, x_5\}, \{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_2, x_6, x_8\}\},$$

$$\{\{0,2,1,2\},\{0,2,1,2\},\{-1,1,1,2\},\{-1,1,0,1\},\{0,0,0,1\},$$
$$\{0,0,1,0\},\{0,2,0,0\},\{2,0,0,0\}\}\Big\}$$

This is the toric data of this Calabi-Yau space. The first and third brackets are the coordinates and the corresponding GLSM charge respectively. The second bracket contains the Stanley-Reisner Ideal obtained by SAGE. The form presented in the list can be directly used to calculate the cohomology in cohomCalg [37, 38].

- Consistent involution and corresponding cohomology.

  The general expression for this part is:

  $$\Big\{\text{Index}, \{\text{Involution 1, Hodge number}\}, \{\text{Involution 2, Hodge number}\}, \dots\Big\}$$

  In this particular example it reads

  $$\Big\{1, \Big\{\{x_4 \to x_5, x_5 \to x_4, x_7 \to x_8, x_8 \to x_7\},$$
  $$\{\{\{1,0,0\},\{0,8,0\},\{0,0,1\}\},\{\{1,3,0\},\{3,6,3\},\{0,3,1\}\}\}\Big\}\Big\}$$

  The index 1 means it is the first space in the list. It contains one type of nontrivial exchange involution $\sigma : \{x_4 \leftrightarrow x_5, x_7 \leftrightarrow x_8\}$ which is consistent with the symmetry of SR-Ideal and exchanges two dP_7 surfaces together with two Wilson surfaces. One can check that this involution is also consistent with the orientifold invariant Kähler cone condition. $\{\{\{1,0,0\},\{0,8,0\},\{0,0,1\}\},\{\{1,3,0\},\{3,6,3\},\{0,3,1\}\}\}$ represents the Hodge number of dP_7 and Wilson divisor respectively.

- Classifications.

  In this part, we record the index of the spaces which appear under the classification according to table 2 and 3. As a result, the index 1 will appear three times in the classification list in $h^{1,1}(CY_3) = 4$, five vertex in the dual-polytope.

4 Typical volume forms for divisor exchange involutions

In this section, we will present some concrete models under each of the classes presented in the section 3. Our main focus will be in obtaining some simple volume forms including the strong/weak swiss-cheese types, so that these spaces could be utilized for model building purpose in (an extended) LARGE volume scenarios (with the inclusion of odd-axions). The examples presented here are mostly for $h^{1,1}(CY_3) = 3$ as we intend to make all the relevant pieces of information available in the article itself. Note that, for spaces with $h^{1,1}(CY_3) = 4$, one has to refer the external file.

4.1 Exchange of completely rigid divisors

First, we consider the Calabi-Yau spaces which have del-Pezzo coordinate divisors (i.e. divisors $D : x_i = 0$ with $h^{1,0}(D) = 0, h^{2,0}(D) = 0$). In addition, these del-Pezzo divisors should satisfy the criteria of being "Nontrivial Identical" which results in 104 spaces. Imposing the SR-Ideal symmetry, this number further reduces to 45 spaces among which, 7 spaces contains pairs of del-Pezzos NIDs while 6 spaces have 'rigid but not del-Pezzos' exchangeable divisors. We consider an example to see the volume form before and after the involution in this class of examples.

Let us consider one example which contains both del-Pezzo and rigid but non-del-Pezzo divisors. This Calabi-Yau threefold is the third example in table 6 and the toric data is

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	-1	-1	0	0	0	1
6	-1	3	2	0	1	1	0
0	2	-2	-1	1	0	0	0
	dP_{14}	dP_{14}	dP_8	dP_8			

with Hodge numbers $(h^{2,1}, h^{1,1}) = (45, 3)$ and Euler number $\chi = -84$. The Stanley-Reisner ideal reads

$$\text{SR} = \{x_1 x_4, x_2 x_3, x_3 x_4, x_1 x_5 x_6 x_7, x_2 x_5 x_6 x_7\}$$

Computing the Hodege diamonds, one find that the divisors $D_{1,2}$ are rigid surfaces dP_{14} while $D_{3,4}$ are del-Pezzo dP_8 surface. The possible involution, which keeps the SR-Ideal symmetry is $\sigma : \{x_1 \leftrightarrow x_2, \& x_3 \leftrightarrow x_4\}$.

Expanding the Kähler form as $J = r^i[K_i]$,, the Kähler cone is given simply by $r^i > 0$. One can show that the Kähler cone is generated by the divisors with following GLSM charges:

$$\{\{0, 1, 0\}, \{-1, 3, -1\}, \{-1, 3, 0\}, \{0, 1, 1\}\} \tag{4.1}$$

Here we see that the number of Kähler cone generators is larger than $h^{1,1}(CY_3)$, this shows that the polytope is non-simplicial.

The triple intersection form under the basis of smooth divisors as $\{D_3, D_4, D_5\}$ reads

$$I_3 = D_3^3 + D_4^3 - D_3^2 D_5 - D_4^2 D_5 + D_3 D_5^2 + D_4 D_5^2 + D_5^3. \tag{4.2}$$

Writing the Kähler form in the above basis of divisors as $J = t_3 D_3 + t_4 D_4 + t_5 D_5$, the resulting volume form in terms of two cycle volumes t_i takes the form

$$\mathcal{V} = \frac{1}{3!} \int_{\mathcal{M}} J \wedge J \wedge J = \frac{1}{6}(t_3^2 + t_4^3 - 3t_3^2 t_5 - 3t_4^2 t_5 + 3t_3 t_5^2 + 3t_4 t_5^2 + t_5^3) \tag{4.3}$$

Then the Kähler cone eq. (4.1) can be expanded under these basis as:

$$K_1 = D_5, \ K_2 = D_3 + D_5, \ K_3 = D_3 + D_4 + D_5, \ K_4 = D_4 + D_5. \tag{4.4}$$

For the Kähler parameters t_i this translates into

$$t_4 > 0, \ t_5 > 0, \ t_3 + t_4 - t_5 > 0 \tag{4.5}$$

Defining the four-cycle volumes $\tau_i = \frac{1}{2} \int_{D_i} J \wedge J$, we find

$$\tau_3 = \frac{1}{2}(t_3 - t_5)^2, \ \tau_4 = \frac{1}{2}(t_4 - t_5)^2, \ \tau_5 = \frac{1}{2}(-t_3^2 - t_4^2 + 2t_3t_5 + 2t_4t_5 + t_5^2). \tag{4.6}$$

Taking into account the Kähler cone constraints, the volume can be written in the strong swiss-cheese form

$$\mathcal{V} = \frac{\sqrt{2}}{9}(\sqrt{3}(\tau_3 + \tau_4 + \tau_5)^{3/2} - 3\tau_3^{3/2} - 3\tau_4^{3/2}) \tag{4.7}$$

The above volume form shows that the large volume limit is given by $\tau_5 \to \infty$ while keeping the other four-cycles small. This also indicates that the dP_8 divisor are shrinkable to a point in Calabi-Yau hypersurface.

Orientifold projection: under the orientifold involution $\sigma : \{x_1 \leftrightarrow x_2, \ x_3 \leftrightarrow x_4\}$, we define $D_\pm = D_3 \pm D_4$ and then the intersection form in the new basis $\{D_\pm, D_5\}$ becomes

$$I_3 = D_5^3 + 2D_5^2 D_+ - 2D_5 D_+^2 + 2D_+^3 - 2D_5 D_-^2 + 2D_+ D_-^2. \tag{4.8}$$

Under orientifold involution, the Kähler form is even, i.e. $\sigma^*(J) = J$ and therefore, it must belong to $H_+^{1,1}(CY_3)$. The involution condition implies that $t_3 = t_4$ and then we can expand the Kähler form in the basis of divisors as $J = t_+ D_+ + t_5 D_5$ and write down the four-cycle volumes τ_i in terms of these two-cycle volumes t_i. Subsequently, the orientifold invariant volume form become

$$\begin{aligned}
\mathcal{V} &= \frac{1}{6}(t_5^3 + 6t_5^2 t_+ - 6t_5 t_+^2 + 2t_+^3) \\
&= \frac{1}{9}(\sqrt{6}(\tau_5 + \tau_+)^{3/2} - 3\tau_+^{3/2})
\end{aligned} \tag{4.9}$$

where

$$\tau_+ = (t_5 - t_+)^2, \ \tau_5 = \frac{1}{2}(t_5^2 + 4t_5 t_+ - 2t_+^2). \tag{4.10}$$

This is a perfect example for the simplest generalization of the standard LARGE volume model with the inclusion of a single involutively odd modulus [18].

4.2 Exchange of Wilson divisors with $h^{1,0}(D) \neq 0$

In this section, we consider the spaces which have Wilson coordinate divisors. In addition, these Wilson divisors should satisfy the criteria of being "Nontrivial Identical" and also involution should keep the SR-Ideal invariant. For $h^{1,1}(CY_3) = 3$, only 4 spaces contain pairs of NIDs with $h^{1,0}(D) \neq 0$. These are numbered as $\{1, 2, 9, 35\}$ in table 6. From the application point of view, we consider example in which the Calabi-Yau volume expression can be written in a simple form. One such example is the second spaces in the table 6 for which the toric data is:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	0	0	-1	-2	0	0	3
0	0	0	-2	-1	0	3	0
6	2	1	1	1	1	0	0
			dP_8	dP_8		W	W

and has Hodge numbers $(h^{2,1}, h^{1,1}) = (45, 3)$ with Euler number $\chi = -84$. The Stanley-Reisner ideal reads

$$\mathrm{SR} = \{x_3 x_4, x_3 x_7, x_4 x_6, x_1 x_2 x_5 x_6, x_1 x_2 x_5 x_7\}$$

After looking at the internal structure, one finds that the divisors $D_{3,4}$ are dP_8 surfaces while the divisors $D_{6,7}$ are 'Wilson' divisor. Here, the Wilson divisor W is characterized by $\{h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}\} \equiv \{1, 4, 0, 2\}$. The possible involution which permutes the two Wilson divisor W and also respects the SR-Ideal symmetry, is defined as; $\sigma : \{x_3 \leftrightarrow x_4 \& x_6 \leftrightarrow x_7\}$.

Expanding the Kähler form as $J = r^i [K_i]$, the Kähler cone defined by $r^i > 0$ is non-simplicial, and it is generated by the divisors with following GLSM charges:

$$\{\{0, 1, 1\}, \{-3, -3, 3\}, \{-2, -1, 2\}, \{-1, -2, 2\}\} \tag{4.11}$$

Let us consider the basis of divisors $\{D_1, D_6, D_7\}$ which is the only basis compatible with an orientifold invariant Kähler cone. The intersection form in this basis reads

$$I_3 = 8D_1^3 - 12D_1 D_6^2 - 24D_6^3 + 6D_1 D_6 D_7 \tag{4.12}$$
$$+ 3D_6^2 D_7 - 12 D_1 D_7^2 + 3D_6 D_7^2 - 24 D_7^3.$$

Writing the Kähler form in the above basis of divisors as $J = t_1 D_1 + t_6 D_6 + t_7 D_7$, the Kähler cone generators eq. (4.11) can be written out as:

$$K_1 = \frac{D_1}{2}, \qquad\qquad K_2 = \frac{3}{2}D_1 - D_6 - D_7, \tag{4.13}$$
$$K_3 = D_1 - \frac{1}{3}D_6 - \frac{2}{3}D_7, \qquad\qquad K_4 = D_1 - \frac{2}{3}D_6 - \frac{1}{3}D_7.$$

For the Kähler parameters t_i this translates into

$$t_1 > 0,\ 3t_1 > 2(t_6 + t_7),\ 3t_1 > t_6 + 2t_7,\ 3t_1 > 2t_6 + t_7. \tag{4.14}$$

Using the intersection polynomial, the volume form in terms of two cycle volumes t_i takes the form

$$\mathcal{V} = \frac{4t_1^3}{3} - 6t_1 t_6^2 - 4t_6^3 + 6t_1 t_6 t_7 + \frac{3t_6^2 t_7}{2} - 6t_1 t_7^2 + \frac{3t_6 t_7^2}{2} - 4t_7^3 \tag{4.15}$$

and subsequently, the corresponding four-cycle volumes can be written as

$$\tau_1 = 4t_1^2 - 6(t_6^2 - t_6 t_7 + t_7^2), \qquad\qquad \tau_6 = -\frac{3}{2}(2t_6 - t_7)(4t_1 + 4t_6 + t_7),$$
$$\tau_7 = -\frac{3}{2}(t_6 - 2t_7)(4t_1 + t_6 + 4t_7). \tag{4.16}$$

It is not possible to write a 'simple' volume form in terms of 4-cycle volumes expressed above. So, we come to the expression after considering the exchange involution.

Orientifold projection: under the orientifold projection $\sigma : \{x_3 \leftrightarrow x_4 \,\&\, x_6 \leftrightarrow x_7\}$, we define $D_\pm = D_6 \pm D_7$ and then the intersection form in the new basis $\{D_1, D_\pm\}$ becomes

$$I_3 = 8D_1^3 - 12D_1D_+^2 - 30D_+^3 - 36D_1D_-^2 - 54D_+D_-^2 \qquad (4.17)$$

Again, the Kähler form is expanded in the orientifold invariant basis $J = t_1 D_1 + t_+ D_+$, and then, the volume form can be written in terms of orientifold invariant 2-cycle volumes and subsequently, in terms of the 4-cycle volume as below

$$\begin{aligned}
\mathcal{V} &= \frac{4t_1^3}{3} - 6t_1t_+^2 - 5t_+^3 \\
&= \frac{1}{18}\left(6\left(\tau_1 + \tau_+\right)^{3/2} + \sqrt{3}\left(3\tau_1 + 2\tau_+\right)^{3/2}\right)
\end{aligned} \qquad (4.18)$$

where

$$\tau_1 = 4t_1^2 - 6t_+^2, \; \tau_+ = -3t_+\left(4t_1 + 5t_+\right) \qquad (4.19)$$

Notice that the aforementioned volume form is not of swiss-cheese type.

In fact, we observe that in all the 4 spaces for $h^{1,1}(CY_3) = 3$ (numbered as $\{1, 2, 9, 35\}$ in table 6) which have involutions defined by the exchange of 'Wilson' divisors, we find that the orientifold invariant volume form expressions take the following forms, $\{D_1, D_\pm = D_6 \pm D_7\}$,

$$\mathcal{V} = a \, t_1^3 - b \, t_1t_+^2 - c \, t_+^3 \qquad (4.20)$$

for some positive values of the constants a, b and c.

4.3 Exchange of deformation divisors with $h^{2,0}(D) \neq 0$

In this section, we consider the spaces which have deformation coordinate divisors (i.e. divisors $D : x_i = 0$ with $h^{2,0}(D) \neq 0$). In addition, such deformation divisor should satisfy the criteria of being "Nontrivial Identical" as well as should respect the SR-Ideal symmetry. There are 35 such spaces in $h^{1,1}(CY_3) = 3$ list in table 6.

Swiss-cheese example. Let us consider the Calabi-Yau space (numbered at 21 in the list) which has the following toric data:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
3	1	1	0	0	0	0	1
3	0	1	0	0	1	1	0
3	1	0	1	1	0	0	0
	SD2	SD2					dP_6

with Hodge numbers $(h^{2,1}, h^{1,1}) = (72, 3)$ and Euler number $\chi = -138$. The Stanley-Reisner ideal are

$$\mathrm{SR} = \{x_1x_7, x_2x_7, x_1x_3x_4, x_2x_5x_6, x_3x_4x_5x_6\}$$

where SD2 type divisor has Hodge diamond as $\{h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}\} \equiv \{1, 0, 2, 30\}$. Also, the divisors $D_{3,4}$ and $D_{5,6}$ are also of nontrivial identical type Hodge diamond data given as $\{h^{0,0}, h^{1,0}, h^{2,0}, h^{1,1}\} \equiv \{1, 0, 1, 23\}$.[7]

The exchange of two SD2 divisors compatible with the SR-Ideal symmetry is given by the involution $\sigma : \{x_1 \leftrightarrow x_2 \,\&\, x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6\}$. Now, expanding the Kähler form as $J = r^i [K_i]$, $i = 1, 2, 3$, the Kähler cone defined via $r^i > 0$ is generated by the divisors with following GLSM charges:

$$\{\{1, 0, 1\}, \{1, 1, 0\}, \{1, 1, 1\}\} \tag{4.21}$$

Focusing on the involution σ, the intersection form in the basis of smooth divisors $\{D_1, D_2, D_7\}$ can be written as

$$I_3 = 3D_1^2 D_2 + 3D_1 D_2^2 + 3D_7^3 \tag{4.22}$$

Writing the Kähler form in the above basis of divisors as $J = t_1 D_1 + t_2 D_2 + t_7 D_7$, the Kähler cone generators eq. (4.21) can be written out as:

$$K_1 = D_1, \; K_2 = D_2, \; K_3 = D_1 + D_2 + D_7 \tag{4.23}$$

which results in the following simple constraints

$$t_1 + t_7 > 0, \; t_2 + t_7 > 0, \; t_7 < 0. \tag{4.24}$$

Using the intersection polynomial, the volume form in terms of two cycle volumes t_i takes the form

$$\mathcal{V} = \frac{3\, t_1^2 t_2}{2} + \frac{3\, t_1 t_2^2}{2} + \frac{t_7^3}{2} \tag{4.25}$$

and subsequently, the corresponding four-cycle volumes can be written as

$$\tau_1 = \frac{3}{2} t_2 (2t_1 + t_2), \; \tau_2 = \frac{3}{2} t_1 (2t_2 + t_1), \; \tau_7 = \frac{3}{2} t_7^2. \tag{4.26}$$

Utilizing the Kähler cone conditions, one can rewrite the volume form in terms of divisor volume expressions as under,

$$\mathcal{V} = \frac{\sqrt{2}}{9} (2\tau_1 - \tau_2 + \beta) \sqrt{-2\tau_1 + \tau_2 + 2\beta} - \frac{\sqrt{2}}{3\sqrt{3}} \tau_7^{3/2}. \tag{4.27}$$

where $\beta = \sqrt{\tau_1^2 - \tau_1 \tau_2 + \tau_2^2}$. Observe that the negative contribution is coming from a del-Pezzo divisor $D_7 = dP_6$. This volume form looks a bit complicated, however, after considering the involution σ, it reduces to a simple and nice form as we will see in a moment.

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[7]This type of deformation divisor does not appear frequently, so we did not take care of such NIDs in our scan.

Orientifold projection: for the orientifold projection $\sigma : \{x_1 \leftrightarrow x_2 \,\&\, x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6\}$, we consider a basis $\{D_\pm = D_1 \pm D_2,\ D_7\}$ in which the intersection polynomial becomes

$$I_3 = 3D_7^3 + 18D_+^3 - 6D_+ D_-^2 \tag{4.28}$$

from which, one can easily deduce the orientifold invariant volume form as under

$$\mathcal{V} = \frac{t_7^3}{2} + 3\,t_+^3 = \frac{1}{9}\left(\tau_+^{3/2} - \sqrt{6}\,\tau_7^{3/2}\right) \tag{4.29}$$

where $\tau_+ = 9\,t_+^2$, $\tau_7 = \frac{3\,t_7^2}{2}$.

K3-fibration example. Let us consider an example in which the volume form reflects the fibration structure as well as the presence of odd moduli. For this, a relevant Calabi-Yau space (numbered at 8 in the list) is given by the following toric data:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
4	2	0	0	0	0	1	1
4	2	1	0	0	1	0	0
4	2	0	1	1	0	0	0
		$K3$	$K3$	$K3$	$K3$	$K3$	$K3$

with Hodge numbers $(h^{2,1}, h^{1,1}) = (115, 3)$ and Euler number $\chi = -224$. The Stanley-Reisner ideal are

$$\mathrm{SR} = \{x_3 x_4,\ x_6 x_7,\ x_1 x_2 x_5\}$$

After checking the SR-Ideal symmetry, the consistent orientifold involution is $\sigma : \{x_3 \leftrightarrow x_6 \,\&\, x_4 \leftrightarrow x_7\}$. Expanding the Kähler form as $J = r^i[K_i]$, $i = 1, 2, 3$, the Kähler cone defined via $r^i > 0$ is generated by the divisors with following GLSM charges:

$$\{\{1, 0, 0\}, \{0, 0, 1\}, \{1, 1, 1\}\} \tag{4.30}$$

Focusing on the involution σ, the intersection form in the basis of smooth divisors $\{D_2, D_3, D_6\}$ can be written as

$$I_3 = 2\,D_2\,D_3\,D_6 \tag{4.31}$$

Writing the Kähler form in the above basis of divisors as $J = t_2 D_2 + t_3 D_3 + t_6 D_6$, the Kähler cone generators eq. (4.30) can be written out as:

$$K_1 = D_6,\ K_2 = D_3,\ K_3 = D_2 + D_3 + D_6 \tag{4.32}$$

implying the Kähler cone conditions $t_6 > 0$, $t_3 > 0$, $t_2 + t_3 + t_6 > 0$. Using the intersection polynomial, the volume form in terms of two cycle volumes t_i takes the form

$$\mathcal{V} = 2\,t_2\,t_3\,t_6 = \frac{\sqrt{\tau_2}\,\sqrt{\tau_3}\,\sqrt{\tau_6}}{\sqrt{2}} \tag{4.33}$$

and subsequently, the corresponding four-cycle volumes can be written as

$$\tau_2 = 2\,t_3\,t_6,\ \tau_3 = 2\,t_2\,t_6,\ \tau_6 = 2\,t_2\,t_3\,.$$

Orientifold projection: for the orientifold projection $\sigma : \{x_3 \leftrightarrow x_6 \,\&\, x_4 \leftrightarrow x_7\}$, we consider a basis $\{D_2, D_\pm = D_3 \pm D_6\}$ in which the intersection polynomial becomes

$$I_3 = 4D_2\,D_+^2 - 4\,D_2\,D_-^2 \tag{4.34}$$

from which, one can easily deduce the volume form as under

$$\mathcal{V} = 2\,t_2\,t_+^2 = \frac{\sqrt{\tau_2}\ \tau_+}{2\sqrt{2}} \tag{4.35}$$

where $\tau_2 = 2\,t_+^2$, $\tau_+ = 4\,t_2\,t_+$. This volume form is extremely simple, however, reflects the interesting features which was intended to be, i.e. K3-fibration structure along with presence of odd moduli in the volume form which will appear after writing out τ_+ in terms of $N = 1$ chiral variables. For obvious reasons, a simplest generalization into a 'weak' swiss-cheese type volume form (useful for the LARGE volume models with a single involutively odd modulus) requires Calabi-Yau spaces with $h^{1,1} = 4$ distributing one modulus appearance for; fibre, base, swiss-cheese hole and the odd modulus. We do not present any $h^{1,1} = 4$ examples as we intend to be simplistic and self-contained within the article from the point of view of availability of topological data. Recall that for $h^{1,1} = 4$ examples, the list is available in a separate external file.

5 $\ H^{1,1}(CY_3/\sigma)$ splitting in non-toric spaces

In this section, we present a special class of examples in which a reflection involution $\sigma : x_i \to -x_i$ can result in nontrivial involutively odd sector with $h^{1,1}_-(CY_3/\sigma) \neq 0$. We will also calculate the representation of cohomology $H^{1,1}(CY_3)$ directly to see under which reflection it will split to odd part and contribute to odd modulus.

As a concrete example, let us consider the Calabi-Yau three-fold expressed as a degree 12 hypersurface in $\mathbb{WCP}^4[1,1,1,3,6]$.[8] The relevant toric data are given as,

	x_1	x_2	x_3	x_4	x_5	x_6
4	2	1	0	0	0	1
12	6	3	1	1	1	0
						$\mathbb{P}^2 \sqcup \mathbb{P}^2$

and has Hodge numbers $(h^{2,1}, h^{1,1}) = (165, 3)$ with Euler number $\chi = -324$. Here, since $h^{1,1} = 3$ exceeds the number of toric equivalence relations, one Kähler deformation is non-toric. The Stanley-Reisner ideal reads

$$\mathrm{SR} = \{x_1 x_2 x_6, x_3 x_4 x_5\}$$

This Calabi-Yau has one divisor D_6 which contains two disjoint pieces $\mathbb{P}^2 \sqcup \mathbb{P}^2$. The Kähler cone condition is just given by $t_2 > 0$ and $t_3 > 0$. The intersection form written in the basis of smooth divisors $\{D_2, D_6\}$ is,

$$I_3 = 18D_2^3 + 18D_6^3 \tag{5.1}$$

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[8]This space is numbered at 1 in the table 7 given in appendix C, and has also been analyzed in [43].

which results in the following volume form,

$$\mathcal{V} \equiv \frac{1}{3!} \int_{\mathcal{M}} J \wedge J \wedge J = 3\,t_2^3 + 3\,t_6^2 = \frac{1}{9}(\tau_2^{3/2} - \tau_6^{3/2}) \tag{5.2}$$

where

$$\tau_2 = 9t_2^2, \ \tau_6 = 9t_6^2.$$

Explicit computation of $h_-^{1,1}(CY_3/\sigma)$. For *non-toric* space the volume form contains less information, it is hard to say under which reflection $h^{1,1}(CY_3)$ will split to odd part. Now, we will calculate the representation for the $H^{1,1}(CY_3)$ to see such splitting explicitly. It can be either written down by anaylzing the Koszul sequence of bundle-valued Cohomology [38] or by the representation of $H^{1,1}(CY_3)$ in terms of homogeneous coordinates [44] for some simple examples. Now, we use the first method to illustrate the procedure.

Since $H^{1,1}(CY_3) = H^1(CY_3; T^*_{CY_3})$, by using the corresponding Koszul sequence for the cotangent bundle (see [37] and references there), the exact sequence becomes

$\mathcal{E}_S^*$	$\mathcal{O}_S(-2,-6) \oplus \mathcal{O}_S(-1,-3)$ $\oplus \mathcal{O}_S(0,-1)^{\oplus 3} \oplus \mathcal{O}_S(-1,0)$	$\mathcal{O}_S^{\oplus 2}$		$\mathcal{O}_S(-4,-12)$	$\mathcal{E}_S^*$	T_S^*
0	0	2		0	0	0
3	1	0	$\Longrightarrow$	0	**3**	**3**
0	0	0		0	0	165
59	61	2		224	59	0
0	0	0		0	0	0

More precisely, the three $(3 = 2+1)$ appearing in $H^1(CY_3; T^*_{CY_3})$ comes from $H^1(\mathcal{E}_S^*)$, in which '2' comes from $H^0(\mathcal{O}_S)$ whose representation is always a constant while '1' comes from $H^1(\mathcal{O}_S(-1,0))$. Again, using the exact sequence in cohomology for the linebundle $\mathcal{O}_S(-1,0)$, we find the '1' contribution to $H^1(\mathcal{O}_S(-1,0))$ coming from $H^2(\mathcal{O}_X(-5,-12))$ on the ambient space X. Assuming the restriction map and embedding map in the sequence to be invariant under involution, the parity of $H^1(CY_3; T^*_{CY_3})$ depends on the parity of $H^2(\mathcal{O}_X(-5,-12))$ under reflection. The polynomial representation of $H^2(\mathcal{O}_X(-5,-12))$ can easily be calculated to be $\frac{1}{x_1 x_2^2 x_6}$. So, only under reflection involution $\sigma : x_1 \leftrightarrow -x_1$ this polynomial changes signs and then, $h^{1,1}(CY_3/\sigma)$ split to $h_+^{1,1} = 2$ and $h_-^{1,1} = 1$. One can also show explicitly that the cohomology of linebundle also splits into $h_+^{0,0}(D_6) = h_-^{0,0}(D_6) = 1$ under such reflection.

In this example, we find that D_6 divisor has two disjoint $\mathbb{P}^2$s which are internally exchanged within D_6 under the reflection involution $\sigma : x_1 \to -x_1$. This is related to the fact that one Kähler deformation is non-toric and there are only two generators of the Kähler cone which is smaller than the number of Kähler moduli $h^{1,1}(CY_3) = 3$. This implies the Kähler form J (which has to be even under the involution σ) can be written in terms of two divisor volumes. Such a case has been also observed in [29] while investigating the zero-mode structure for generating poly-instanton corrections.

6 Conclusions

In this paper, we presented a classification of the toric Calabi-Yau threefolds with $h^{1,1} \leq 4$ relevant for nontrivial involutively odd sector of (1,1)-cohomology class. For this purpose, we studied two types of involutions; the first one permutes "Nontrivial Identical Divisor(NID)" under exchange of coordinates $\sigma : x_i \leftrightarrow x_j$ while the other one is a reflection $\sigma : x_i \leftrightarrow -x_i$ on *non-toric* space. Both of these types of involutions can result in a non-zero $h^{1,1}_-(CY_3/\sigma)$ for the Calabi-Yau threefolds. In our scanning, we search for the spaces with NIDs and find that imposing SR-Ideal symmetry on the divisor exchange involutions dramatically decreases number of suitable spaces. As a result, we get the list of toric data for the Calabi-Yau threefold with $h^{1,1} \leq 4$ which contains non-zero $h^{1,1}_-(CY_3/\sigma)$ when we restrict to *non-toric* spaces and exchange involution for completely rigid divisors, Wilson divisors and three kinds of deformation divisors. Such explicit constructions of Calabi Yau orientifolds provide a suitable background to extend the type IIB setups (such as KKLT or LARGE volume scenarios) with the inclusion of odd-moduli in the spectrum, and can be useful for concrete as well as promising model building in particle phenomenology and string cosmology. It is straightforward to generalize our result to include more spaces as Calabi-Yau threefolds with $h^{1,1} \geq 5$ as well as Calabi-Yau fourfolds. It would also be interesting to generalize our method to study the spaces beyond the Kreuzer-Skarke list, like higer dimensional polytopes and CICY manifolds (for the list, see [45, 46]).

A Computation tools

There are several packages which can analyze some properties of toric varieties, but sometimes it is not convenient for each of them to do a complete calculation. For example, sometimes PALP fails to triangulate the polytope by itself. This happens whenever the dimension of the polytope is different from four, or when the polytope contains points which are interior to facets. The later one may not cause problem if we just consider the HodgeDiamond of the coordinate divisor on a Calabi-Yau hypersurface since these interior points divisor don't intersect with the Calabi-Yau hypersurfaces, so these correspond to null HodgeDiamond. However, if we want to keep control over the lattice points involved in the triangulation, especially when we analyze general non-Calabi-Yau hypersurface, it is important to triangulate these points fully. In fact, this is exactly what SAGE can do. However, it is very hard for SAGE itself to do some scanning stuff. For getting the intersection number and intersection form SINGULAR is much more efficient. On the scanning

No.	Toric data
	GLSM : $[(1,0),(1,0),(1,0),(0,1),(0,1),(0,1)]$
1	SR : $\{x_1 x_2 x_3, x_4 x_5 x_6\}$ and $x_i \equiv SD2 \quad \forall i.$
	$\sigma : \{x_1 \leftrightarrow x_4, x_2 \leftrightarrow x_5, x_3 \leftrightarrow x_6\}$

Table 5. List of CY_3 spaces with $h^{1,1} = 2$ for the possibility of $h^{1,1}_-(CY_3/\sigma) \neq 0$ under divisor exchange involutions.

issue, some previous program can only contain one of many maximal triangulations and more importantly, it is a black-box which is hard to change and extend. So, it is worthwhile to combine these package in a systematic way which can share some advantages of them and avoid complicate data transfer across several packages.

In this compact scanning program [36], we try to combine some properties of SAGE, PALP(SINGULAR) into a single Mathematica file based on python language. There are two similar scanning procedures depending on different forms of input: one is scanning the GLSM charges as input and the other one is scanning the data from Kreuzer-Skarke list. For the later one, we collect the vertex data in the dual-polytope after blow-up from PALP, then put these data as input to SAGE for fully triangulations. Here these vertex data can get collected in two different ways. One is to include the points which are interior to facets and the other one is ignoring them as PALP did itself. After triangulations, we can go further to get the Kähler cone generators in terms of line-bundle charges and meanwhile utilizing these data back to PALP(SINGULAR) to get the triple intersection number and intersection form. Moreover, this analyze can also be performed beyond the Kreuzer-Skarke list like higer dimensional polytopes and CICY manifolds (for the list, see [45, 46]).

B   List of CY_3 for $h^{1,1}_- \neq 0$ under divisor exchange involutions

In this section, we provide the toric data relevant for those Calabi-Yau spaces in which nontrivial identical 'coordinate' divisors have been found in the scan. Note, that we have investigated only coordinate divisors for "Nontrivial Identical" criteria and hence, it is essential for us to provide the explicit GLSM charges along with the SR-Ideal for each space. It is understood that our scan does not capture all divisors which could be "Nontrivial Identical"; e.g. those which may come from a combination of GLSM charges of various coordinate divisors.

Here we present the list of results for $h^{1,1}(CY_3) = 2$ and 3. For $h^{1,1} = 4$ please see the external Mathematica file "Classification of Calabi-Yau Threefolds with Divisor Exchange Involution".

$h^{1,1}(CY_3) = 2$

In the scan of 39 CalabiYaus with $h^{1,1} = 2$, we find that only one has "Nontrivial Identical" coordinate divisors. The toric data for this Calabi-Yau is given in table 5 along with the possible exchange involution.

$h^{1,1}(CY_3) = 3$

In the collection of the Calabi-Yau threefolds with $h^{1,1}(CY_3) = 3$ below, we have seven charge vectors in GLSM corresponding to each coordinate x_i for $i \in \{1, 2, \ldots, 7\}$ along with the SR-Ideal[9]

No.	Toric data
1	GLSM : $[(4,0,4),(1,-1,1),(1,-1,1),(0,0,1),(0,0,1),(0,2,0),(2,0,0)]$
	SR : $\{x_2x_3, x_4x_5, x_1x_6x_7\}$
	$\sigma_1 : \{x_2 \leftrightarrow x_4 \,\&\, x_3 \leftrightarrow x_5 \equiv K3\}; \quad \sigma_2 : x_6 \leftrightarrow x_7 \equiv \{1,3,0,2\}; \sigma_3 = \sigma_1 \cup \sigma_2$
	Kähler Cone generators (KC): $\{(0,0,1),(1,-1,1),(2,0,2)\}$
2	GLSM : $[(0,0,2),(0,0,1),(-1,-2,1),(-2,-1,1),(0,0,1),(0,3,0),(3,0,0)]$
	SR : $\{x_3x_4, x_3x_7, x_4x_6, x_1x_2x_5x_6, x_1x_2x_5x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_4 \equiv dP_8 \,\&\, x_6 \leftrightarrow x_7 \equiv \{1,4,0,2\}\}$
	KC: $\{(0,0,1),(-3,-3,3),(-2,-1,2),(-1,-2,2)\}$
3	GLSM : $[(1,-1,2),(-1,3,-2),(-1,2,-1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_4, x_2x_3, x_3x_4, x_1x_5x_6x_7, x_2x_5x_6x_7\}$
	$\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_{14} \,\&\, x_3 \leftrightarrow x_4 \equiv dP_8\}$
	KC: $\{(0,1,0),(-1,3,-1),(-1,3,0),(0,1,1)\}$

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[9]After considering all possible maximal triangulations, there is a possibility that some spaces coming from different polytopes might be repeated. For example, we can see this situation explicitly in $h^{11} = 3$ where spaces numbered as 5-8 are the same as those numbered as 39-42. However, in order to use the labeling of spaces consistent with the external file, we count all such spaces.

No.	Toric data
4	GLSM : $[(-2, 2, 1), (0, 0, 1), (1, 1, 0), (0, 1, 0), (0, 1, 0), (1, 0, 0), (1, 0, 0)]$
	SR : $\{x_1x_2, x_1x_4x_5, x_3x_4x_5, x_3x_6x_7, x_2x_6x_7\}$
	$\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_1 \ \& \ x_4 \leftrightarrow x_6 \ \& \ x_5 \leftrightarrow x_7 \equiv \{1, 0, 1, 21\}\}$
	KC: $\{(1, 1, 0), (2, 2, 1), (0, 2, 1), (0, 4, 1)\}$
5 (39)	GLSM : $[(2, 4, 2), (1, 2, 0), (0, 0, 1), (0, 0, 1), (0, 1, 0), (0, 1, 0), (1, 0, 0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma_1 : \{x_3 \leftrightarrow x_5 \ \& \ x_4 \leftrightarrow x_6 \equiv K3\} \quad$ and $\quad x_7 \equiv W$
	KC: $\{(0, 1, 0), (0, 0, 1), (1, 2, 1)\}$
6-8 (40-42)	GLSM : $[(2, 2, 2), (0, 1, 0), (0, 0, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 0, 0)]$
	SR : $\{x_2x_5, x_3x_4, x_1x_6x_7\}$
	$\sigma : \{x_2 \leftrightarrow x_3 \ \& \ x_4 \leftrightarrow x_5\}, \qquad x_i \equiv K3 \ \forall i = \{2, \dots, 7\}$
	KC : $\{(0, 0, 1), (0, 1, 0), (1, 1, 1)\}$
	SR : $\{x_2x_5, x_6x_7, x_1x_3x_4\}$
	$\sigma : \{x_2 \leftrightarrow x_6 \ \& \ x_5 \leftrightarrow x_7\}, \qquad x_i \equiv K3 \ \forall i = \{2, \dots, 7\}$
	KC : $\{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$
	SR : $\{x_3x_4, x_6x_7, x_1x_2x_5\}$
	$\sigma : \{x_3 \leftrightarrow x_6 \ \& \ x_4 \leftrightarrow x_7\} , \qquad x_i \equiv K3 \ \forall i = \{2, \dots, 7\}$
	KC: $\{(1, 0, 0), (0, 0, 1), (1, 1, 1)\}$
9	GLSM : $[(2, 0, 2), (0, 0, 1), (1, -1, 1), (1, -1, 1), (0, 0, 1), (0, 2, 0), (2, 0, 0)]$
	SR : $\{x_2x_5, x_3x_4, x_1x_6x_7\}$
	$\sigma_1 : \{x_2 \leftrightarrow x_3 \ \& \ x_4 \leftrightarrow x_5 \equiv K3\}; \sigma_2 : x_6 \leftrightarrow x_7 = \{1, 4, 0, 2\}; \sigma_3 = \sigma_1 \cup \sigma_2$
	KC: $\{(0, 0, 1), (1, -1, 1), (2, 0, 2)\}$
10	GLSM : $[(0, 0, 1), (1, 2, -1), (1, 2, -1), (0, 0, 1), (0, 1, 0), (0, 1, 0), (1, 0, 0)]$
	SR : $\{x_1x_4, x_5x_6, x_2x_3x_7\}$
	$\sigma : \{x_1 \leftrightarrow x_5 \ \& \ x_4 \leftrightarrow x_6 \equiv K3\};$ KC: $\{(0, 1, 0), (0, 0, 1), (1, 2, 0)\}$
11	SR : $\{x_2x_3, x_5x_6, x_1x_4x_7\}$
	$\sigma : \{x_2 \leftrightarrow x_5 \ \& \ x_3 \leftrightarrow x_6 \equiv K3\};$ KC: $\{(0, 1, 0), (1, 2, -1), (1, 2, 0)\}$
12	GLSM : $[(1, 0, 1), (0, 1, 1), (-1, -1, 1), (0, 0, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0)]$
	SR : $\{x_1x_7, x_2x_6, x_3x_4x_5\} \qquad x_3 \equiv dP_9$
	$\sigma : \{x_1 \leftrightarrow x_2 \equiv \{1, 0, 3, 37\} \ \& \ x_6 \leftrightarrow x_7 \equiv dP_8\}$
	KC: $\{(0, 1, 1), (1, 0, 1), (0, 0, 1)\}$
13	GLSM : $[(-1, 1, 1), (0, 0, 1), (2, -1, 0), (0, 1, 0), (1, 0, 0), (1, 0, 0), (1, 0, 0)]$
	SR : $\{x_1x_2, x_1x_4, x_2x_3, x_3x_5x_6x_7, x_4x_5x_6x_7\}$
	$\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_7 \ \& \ x_3 \leftrightarrow x_4 \equiv dP_{20}\};$ KC:$\{(1, 0, 0), (0, 1, 2), (1, 0, 1), (0, 1, 1)\}$
14	GLSM : $[(1, 2, -1), (1, 2, 0), (0, 0, 1), (0, 0, 1), (0, 1, 0), (0, 1, 0), (1, 0, 0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_5 \ \& \ x_4 \leftrightarrow x_6 \equiv K3\};$ KC: $\{(0, 1, 0), (0, 0, 1), (1, 2, 0)\}$
15	GLSM : $[(-1, 1, 1), (0, 0, 1), (1, 1, 0), (0, 1, 0), (0, 1, 0), (1, 0, 0), (1, 0, 0)]$
	SR : $\{x_1x_2, x_1x_4x_5, x_3x_4x_5, x_3x_6x_7, x_2x_6x_7\}$
	$\sigma : x_1 \leftrightarrow x_2 \equiv dP_{11};$ KC: $\{(1, 1, 0), (1, 1, 1), (0, 1, 1), (0, 2, 1)\}$

No.	Toric data
16	GLSM : $[(-1,1,1),(0,1,0),(0,0,1),(0,0,1),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_2x_5, x_6x_7, x_1x_3x_4\}$ $\sigma_1 : \{x_2 \leftrightarrow x_6 \,\&\, x_5 \leftrightarrow x_7 \equiv K3\}$, and $x_1 \equiv dP_{19}$ KC: $\{(0,1,0),(1,0,0),(0,1,1)\}$
17	SR : $\{x_3x_4, x_6x_7, x_1x_2x_5\}$ $\sigma_1 : \{x_3 \leftrightarrow x_6 \,\&\, x_4 \leftrightarrow x_7 \equiv K3\}$, and $x_1 \equiv dP_{19}$ KC: $\{(0,0,1),(1,0,0),(0,1,1)\}$
18	GLSM : $[(-1,1,0),(-1,0,1),(0,0,1),(0,1,0),(2,0,0),(1,0,0),(1,0,0)]$
	SR : $\{x_1x_4, x_2x_3, x_5x_6x_7\}$ $\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_8 \,\&\, x_3 \leftrightarrow x_4 \equiv \{1,0,2,29\},\ x_1 \leftrightarrow x_2\}$ KC: $\{(0,0,1),(0,1,0),(1,0,0)\}$
19	GLSM : $[(-1,1,0),(-1,0,1),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_2, x_1x_5x_6, x_2x_3x_4, x_3x_4x_7, x_5x_6x_7\}$ $\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_{10} \,\&\, x_3 \leftrightarrow x_4 \,\&\, x_5 \leftrightarrow x_6 \equiv SD2\}$ KC: $\{(0,0,1),(0,1,0),(-1,1,1)\}$
20	GLSM : $[(-1,0,1),(0,-1,1),(0,0,1),(0,1,0),(1,0,0),(0,1,0),(1,0,0)]$
	SR : $\{x_4x_6, x_5x_7, x_1x_2x_3\}$ $\sigma_1 : x_1 \leftrightarrow x_2 \equiv dP_{16};\quad \sigma_2 : \{x_4 \leftrightarrow x_5 \,\&\, x_6 \leftrightarrow x_7 \equiv K3\};\ \sigma_3 = \sigma_1 \cup \sigma_2$ KC: $\{(0,1,0),(1,0,0),(0,0,1)\}$
21	GLSM : $[(1,0,1),(1,1,0),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_7, x_2x_7, x_1x_3x_4, x_2x_5x_6, x_3x_4x_5x_6\}$ $\sigma : \{x_1 \leftrightarrow x_2 \equiv SD2 \,\&\, x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv \{1,0,1,23\}\}\qquad x_7 \equiv dP_6$ KC: $\{(1,0,1),(1,1,0),(1,1,1)\}$
22	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$ $\sigma_1 : \{x_1 \leftrightarrow x_2 \equiv SD2\};\quad \sigma_2 : \{x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv K3\};\quad \sigma_3 = \sigma_1 \cup \sigma_2$ KC: $\{(0,0,1),(0,1,0),(1,1,1)\}$
23	GLSM : $[(0,0,1),(0,0,1),(1,2,0),(1,2,0),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_2, x_5x_6, x_3x_4x_7\}$ $\sigma : \{x_1 \leftrightarrow x_5 \,\&\, x_2 \leftrightarrow x_6 \equiv K3\}$ KC: $\{(0,0,1),(0,1,0),(1,2,0)\}$
24	GLSM : $[(0,0,1),(0,0,1),(1,1,0),(0,1,0),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_1x_2, x_4x_5, x_3x_6x_7\}$ $\sigma : \{x_1 \leftrightarrow x_4 \,\&\, x_2 \leftrightarrow x_5 \equiv K3\}$, KC: $\{(0,0,1),(0,1,0),(1,1,0)\}$
25	SR : $\{x_1x_2, x_6x_7, x_3x_4x_5\}$ $\sigma : \{x_1 \leftrightarrow x_6 \,\&\, x_2 \leftrightarrow x_7 \equiv K3\}$, KC: $\{(0,0,1),(1,0,0),(1,1,0)\}$
26	GLSM : $[(0,0,1),(0,0,1),(-1,1,0),(0,1,0),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_1x_2, x_6x_7, x_3x_4x_5\}$ $\sigma : \{x_1 \leftrightarrow x_6 \,\&\, x_2 \leftrightarrow x_7 \equiv K3\}$ and $x_3 \equiv dP_{13}$ KC: $\{(0,0,1),(0,1,0),(1,0,0)\}$
27	GLSM : $[(0,1,0),(0,0,1),(0,0,1),(0,1,0),(1,0,0),(1,0,0),(1,0,0)]$
	SR : $\{x_1x_4, x_2x_3, x_5x_6x_7\}$ $\sigma : \{x_1 \leftrightarrow x_2 \,\&\, x_3 \leftrightarrow x_4 \equiv K3\}$ and $x_5 = x_6 = x_7 \equiv SD2$ KC: $\{(1,0,0),(0,0,1),(0,1,0)\}$

No.	Toric data
28	GLSM : $[(0,1,1),(-1,1,0),(0,0,1),(0,0,1),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_3x_4, x_6x_7, x_1x_2x_5\}$
	$\sigma : \{x_3 \leftrightarrow x_6 \,\&\, x_4 \leftrightarrow x_7 \equiv K3\}$ and $x_2 \equiv dP_{10}$
	KC: $\{(1,0,0),(0,0,1),(0,1,1)\}$
29	GLSM : $[(-1,1,0),(-1,0,1),(0,0,1),(0,1,0),(1,0,0),(1,0,0),(1,0,0)]$
	SR : $\{x_1x_4, x_2x_3, x_5x_6x_7\}$
	$\sigma : \{x_1 \leftrightarrow x_2 \equiv dP_7 \,\&\, x_3 \leftrightarrow x_4 \equiv \{1,0,3,38\}\}$ $x_5 = x_6 = x_7 \equiv SD2$
	KC: $\{(0,0,1),(0,1,0),(1,0,0)\}$
30	GLSM : $[(1,2,1),(1,2,0),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv K3\}$ and $x_7 \equiv W$
	KC: $\{(0,1,0),(0,0,1),(1,2,1)\}$
31	GLSM : $[(-1,-1,1),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_4x_5, x_6x_7, x_1x_2x_3\}$
	$\sigma : \{x_4 \leftrightarrow x_6 \,\&\, x_5 \leftrightarrow x_7 \equiv K3\}$ and $x_1 \equiv dP_7$
	KC: $\{(1,0,0),(0,1,0),(0,0,1)\}$
32-34	GLSM : $[(1,1,1),(0,1,0),(0,0,1),(0,0,1),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_2x_5, x_3x_4, x_1x_6x_7\}$
	$\sigma : \{x_2 \leftrightarrow x_3 \,\&\, x_4 \leftrightarrow x_5\};$ $x_i \equiv K3 \,\forall i = \{2,\dots,7\}$
	KC: $\{(0,0,1),(0,1,0),(1,1,1)\}$
	SR : $\{x_2x_5, x_6x_7, x_1x_3x_4\}$
	$\sigma : \{x_2 \leftrightarrow x_6, \,\&\, x_5 \leftrightarrow x_7\},$ $x_i \equiv K3 \,\forall i = \{2,\dots,7\}$
	KC: $\{(1,0,0),(0,1,0),(1,1,1)\}$
	SR : $\{x_3x_4, x_6x_7, x_1x_2x_5\}$
	$\sigma : \{x_3 \leftrightarrow x_6, \,\&\, x_4 \leftrightarrow x_7\},$ $x_i \equiv K3 \,\forall i = \{2,\dots,7\}$
	KC: $\{(1,0,0),(0,0,1),(1,1,1)\}$
35	GLSM : $[(0,4,4),(0,0,1),(-1,1,1),(-1,1,1),(0,0,1),(0,2,0),(2,0,0)]$
	SR : $\{x_2x_5, x_3x_4, x_1x_6x_7\}$
	$\sigma_1 : x_6 \leftrightarrow x_7 \equiv \{1,3,0,2\};\; \sigma_2 : \{x_2 \leftrightarrow x_3 \,\&\, x_4 \leftrightarrow x_5 \equiv K3\}, \sigma_3 = \sigma_1 \cup \sigma_2$
	KC: $\{(-1,1,1),(0,0,1),(0,2,2)\}$
36	GLSM : $[(1,2,1),(1,2,1),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv K3\}$ and $x_7 \equiv dP_1$
	KC: $\{(0,1,0),(0,0,1),(1,2,1)\}$
37	GLSM : $[(2,4,1),(1,2,-1),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv K3\}$
	KC: $\{(0,1,0),(0,0,1),(2,4,1)\}$
38	GLSM : $[(1,3,2),(-1,1,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_2x_3, x_1x_4x_5, x_1x_6x_7, x_2x_4x_5, x_3x_6x_7\}$
	$\sigma : \{x_2 \leftrightarrow x_3 \equiv dP_{13} \,\&\, x_4 \leftrightarrow x_6 \equiv K3 \,\&\, x_5 \leftrightarrow x_7 \equiv K3\}$
	KC: $\{(1,3,2),(0,1,1),(1,3,3),(0,4,3)\}$

No.	Toric data
43	GLSM : $[(1,1,2),(-1,-1,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0),(1,0,0)]$
	SR : $\{x_4x_5, x_6x_7, x_1x_2x_3\}$
	$\sigma : \{x_4 \leftrightarrow x_6 \,\&\, x_5 \leftrightarrow x_7 \equiv K3\}$
	KC: $\{(1,0,0),(0,1,0),(1,1,2)\}$
44	GLSM : $[(2,4,3),(1,2,1),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_5 \,\&\, x_4 \leftrightarrow x_6 \equiv K3\}$
	KC: $\{(0,1,0),(0,0,1),(2,4,3)\}$
45	GLSM : $[(3,6,6),(2,4,4),(0,0,1),(0,1,0),(0,0,1),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_5, x_4x_6, x_1x_2x_7\}$
	$\sigma : \{x_3 \leftrightarrow x_4 \,\&\, x_5 \leftrightarrow x_6 \equiv K3\}$ and $x_7 \equiv dP_1$
	KC: $\{(0,1,0),(0,0,1),(1,2,2)\}$

Table 6. List of CY_3 spaces with $h^{1,1} = 3$ for the possibility of $h^{1,1}_-(CY_3/\sigma) \neq 0$ under divisor exchange involutions.

C   List of non-toric spaces

The spaces presented in this section are special examples in which the reflection involution can also result in $h^{1,1}_-(CY_3/\sigma) \neq 0$. The special thing about these spaces is the fact that in each case, a single divisor itself has several disjoint $\mathbb{P}^2$s or two disjoint dP_1s, and under some reflection involution $x_i \to -x_i$, these are exchanged within the divisor itself.

No.	Toric data
1	GLSM : $[(2,6),(1,3),(0,1),(0,1),(0,1),(1,0)]$
	SR : $\{x_1x_2x_6, x_3x_4x_5\}$, $x_6 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 2_+ + 1_-$
2	GLSM : $[(2,4,4),(1,2,2),(0,0,1),(0,1,0),(0,0,1),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_5, x_4x_6, x_1x_2x_7\}$, $x_7 \equiv dP_1 \sqcup dP_1$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$
3	GLSM : $[(2,4,8),(1,2,4),(0,1,2),(0,0,1),(0,0,1),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_6, x_4x_5, x_1x_2x_7\}$, $x_7 \equiv dP_1 \sqcup dP_1$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$
4	GLSM : $[(1,3),(1,3),(0,1),(0,1),(0,1),(1,0)]$
	SR : $\{x_1x_2x_6, x_3x_4x_5\}$, $x_6 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$ or $\sigma_2 : x_2 \leftrightarrow -x_2$, $h^{1,1} = 3_+ + 1_-$
	$\sigma_3 : \{x_1 \leftrightarrow -x_1 \,\&\, x_2 \leftrightarrow -x_2\}$, $h^{1,1} = 2_+ + 2_-$
5	GLSM : $[(-1,3,9),(-1,1,3),(0,0,2),(0,0,2),(0,0,2),(0,2,0),(2,0,0)]$
	SR : $\{x_1x_2, x_6x_7, x_3x_4x_5\}$, $x_6 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$

6-7	GLSM : $[(1,3,2),(0,1,1),(0,1,1),(0,0,1),(0,0,1),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_6, x_1x_7, x_2x_3x_5, x_4x_5x_7, x_2x_3x_4x_5\}$, $\quad x_7 \equiv dP_1 \sqcup dP_1$
	$\sigma_1 : x_4 \leftrightarrow -x_4$, $\sigma_2 : x_5 \leftrightarrow -x_5$, $h^{1,1} = 3_+ + 1_-$
	SR : $\{x_1x_7, x_4x_5, x_2x_3x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_4 \leftrightarrow -x_4$, $\sigma_2 : x_5 \leftrightarrow -x_5$, $h^{1,1} = 3_+ + 1_-$
8	GLSM : $[(1,2,1),(0,-1,1),(0,-1,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_7, x_1x_5x_6, x_2x_3x_4, x_2x_3x_7, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_2 \leftrightarrow -x_2$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
9	GLSM : $[(1,3,0),(0,0,1),(0,0,1),(0,1,0),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_7, x_2x_3, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_2 \leftrightarrow -x_2$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
10	GLSM : $[(1,3,1),(1,3,-1),(0,0,1),(0,1,0),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_3, x_2x_7, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$
11	GLSM : $[(-1,-3,1),(0,0,1),(1,3,0),(0,1,0),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_1x_2, x_3x_7, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$, $\sigma_2 : x_2 \leftrightarrow -x_2$, $h^{1,1} = 3_+ + 1_-$
12-13	GLSM : $[(2,1,5),(1,1,2),(0,-1,1),(0,0,1),(0,0,1),(0,1,0),(1,0,0)]$
	SR : $\{x_2x_6, x_1x_2x_7, x_1x_3x_7, x_3x_4x_5, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
	SR : $\{x_3x_7, x_1x_2x_6, x_1x_2x_7, x_3x_4x_5x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
14-15	GLSM : $[(2,3,3),(1,2,1),(0,-1,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_1x_2x_7, x_1x_3x_7, x_2x_5x_6, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
	SR : $\{x_2x_7, x_1x_3x_4, x_1x_3x_7, x_2x_5x_6, x_4x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma_1 : x_1 \leftrightarrow -x_1$, $\sigma_2 : x_3 \leftrightarrow -x_3$, $h^{1,1} = 3_+ + 1_-$
16-17	GLSM : $[(2,6,4),(1,3,2),(0,1,1),(0,0,1),(0,1,0),(0,1,0),(1,0,0)]$
	SR : $\{x_3x_4, x_4x_7, x_1x_2x_7, x_3x_5x_6x_1x_2x_5x_6\}$, $\quad x_7 \equiv \mathbb{P}^2 \sqcup \mathbb{P}^2$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$
	SR : $\{x_3x_4, x_5x_6, x_1x_2x_7\}$, $\quad x_7 \equiv dP_1 \sqcup dP_1$
	$\sigma : x_1 \leftrightarrow -x_1$, $h^{1,1} = 3_+ + 1_-$

Table 7. List of *non-toric* CY_3 spaces with the possibility of $h^{1,1}_-(CY_3/\sigma) \neq 0$ with reflection involutions. In all these examples, $h^{1,1}(CY_3)$ is more than the equivalence relations for GLSM charges, and hence one Kähler deformation is *non-toric*.

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