A Model-Independent Discussion of Quark Number Density and Quark Condensate at Zero Temperature and Finite Quark Chemical Potential

2015 Chinese Phys. Lett. 32 121101

(http://iopscience.iop.org/0256-307X/32/12/121101)

A Model-Independent Discussion of Quark Number Density and Quark Condensate at Zero Temperature and Finite Quark Chemical Potential *

XU Shu-Sheng(徐书生)[1,4], JIANG Yu(蒋宇)[2], SHI Chao(史潮)[1,4], CUI Zhu-Fang(崔著钫)[1,4], ZONG Hong-Shi(宗红石)[1,3,4]**

[1]Department of Physics, Nanjing University, Nanjing 210093
[2]Center for Statistical and Theoretical Condensed Matter Physics, Zhejiang Normal University, Jinhua 321004
[3]Joint Center for Particle, Nuclear Physics and Cosmology, Nanjing 210093
[4]State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190

(Received 17 August 2015)

Generally speaking, the quark propagator is dependent on the quark chemical potential in the dense quantum chromodynamics (QCD). By means of the generating functional method, we prove that the quark propagator actually depends on $p_4 + i\mu$ from the first principle of QCD. The relation between quark number density and quark condensate is discussed by analyzing their singularities. It is concluded that the quark number density has some singularities at certain μ when $T = 0$, and the variations of the quark number density as well as the quark condensate are located at the same point. In other words, at a certain μ the quark number density turns to nonzero, while the quark condensate begins to decrease from its vacuum value.

PACS: 11.30.Rd, 25.75.Nq, 12.38.Mh, 12.39.−x DOI: 10.1088/0256-307X/32/12/121101

The properties of quantum chromodynamics (QCD) at finite temperature T and finite quark chemical potential μ have been of fundamental interest for decades. There have also been numerous heavy ion collision experiments performed on facilities like the famous RHIC and LHC. In particular, the equation of state at low T and high μ is also very important for astrophysics studies of compact stars.[1]

Nowadays, it is widely believed that there is phase transition from a chiral symmetry broken phase to a chiral symmetry restoration phase during the increase of T and/or μ, and the study of QCD phase diagram is still a long-lived issue. As is well known, enhanced fluctuations are an essential characteristic of QCD phase transitions. In the confined/chirally broken phase, charges are related to hadrons with integer units, and in the deconfined/chirally restored phase, charges are related to quarks with fraction units. Therefore the charge fluctuations which could be associated with the corresponding QCD susceptibilities are different in the two phases and can be used to identify the formation of the quark gluon plasma (QGP). A particularly important one among these charges is the quark number density which has been extensively studied in recent years.[2−6] On the other hand, in the chiral limit case (the current quark mass m=0) the quark condensate can be regarded as the corresponding order parameter and then plays a key role in the study of the chiral restoration phase transition.[7−11] There are many effective models used to discuss quark number density and chiral phase transition, such as the Nambu–Jona–Lasinio (NJL) model,[12−14] Dyson–Schwinger equations.[15−20] In this Letter, we analyze the quark number density and the quark condensate at zero temperature and finite quark chemical potential, and prove that the two quantities begin to change at the same point.

Let us start with the quark number density $\rho(\mu, T)$, which can be defined at finite T and finite μ (in the present study we only work with u and d quarks, and ignore the flavor mixing between u, d quarks in the vector channel) as follows

$$\rho(\mu, T) = \frac{T}{V}\frac{\partial \ln Z(\mu, T)}{\partial \mu}, \tag{1}$$

where V is the spatial volume, and $Z(\mu, T)$ is the partition function of QCD at finite μ and T. From Eq. (1) one can easily derive the following expression of the quark number density with functional integral techniques (we would confine ourselves to the zero temperature case),

$$\rho(\mu) = -N_c N_f Z_2(\zeta^2, \Lambda^2)\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\{S(q,\zeta)[\mu]\gamma_4\}, \tag{2}$$

where ζ is the renormalization point, Λ is the regularization mass scale, $Z_2(\zeta^2, \Lambda^2)$ is the quark wave function renormalization constant, and $S(q, \zeta)[\mu]$ is the dressed quark propagator at finite μ. The trace operation is over Dirac indices, and N_c and N_f are the number of colors and flavors, respectively. From Eq. (2) one can find that the quark number density $\rho(\mu)$ is fully determined by the dressed quark propagator $S(q, \zeta)[\mu]$.

*Supported by the National Natural Science Foundation of China under Grant Nos 11275097, 11475085, 11105122, and 11535005, and the Jiangsu Planned Projects for Postdoctoral Research Funds under Grant No 1402006C.
**Corresponding author. Email: zonghs@nju.edu.cn

In the chiral limit case and $\mu = 0$, the gauge-invariant expression for the renormalization-point-dependent vacuum quark condensate[21] is defined as

$$-\langle\bar{q}q\rangle_\zeta = N_c N_f Z_4(\zeta^2, \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{S(q,\zeta)\}, \quad (3)$$

where $Z_4(\zeta^2, \Lambda^2) = Z_m(\zeta^2, \Lambda^2) Z_2(\zeta^2, \Lambda^2)$ with $Z_m(\zeta^2, \Lambda^2)$ being the mass renormalization constant. At finite μ, the above definition can be generalized to

$$-\langle\bar{q}q\rangle_\zeta[\mu] = N_c N_f Z_4(\zeta^2, \Lambda^2) \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{S(q,\zeta)[\mu]\}. \quad (4)$$

From Eq. (4) it can be seen that the quark condensate is also fully determined by the dressed quark propagator $S(q,\zeta)[\mu]$. Comparing Eqs. (2) and (4) one can find that the dressed quark propagator at finite μ plays a key role in determining the quark number density and the quark condensate. The quark number susceptibility is defined as[22−24]

$$\chi = \frac{\partial\rho}{\partial\mu}. \quad (5)$$

Then one can find the following expression (at the zero temperature case, the detailed discussion can be found in Ref. [25])

$$\chi = N_f Z_2(\zeta^2, \Lambda^2) \int d^4y \langle 0|T\bar{q}(0)\gamma_4 q(0)\bar{q}(y)\gamma_4 q(y)|0\rangle_\mu, \quad (6)$$

where q is the quark field and $|0\rangle$ is the vacuum state. The subscript μ means that the average is taken at the finite quark chemical potential μ. With the external field method, one can obtain the expression of χ as follows:

$$\chi = -N_f N_c Z_2(\zeta^2, \Lambda^2) \times \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\{S(p)[\mu]\Gamma_4(p,0)[\mu]S(p)[\mu]\gamma_4\}, \quad (7)$$

where $\Gamma_4(k,p)[\mu]$ is the fourth part of $\Gamma_\nu(k,p)[\mu]$, which is the dressed vector vertex at finite μ with k being the relative and p being the total momentum of quark–antiquark pair.

When $\mu = 0$, the so-called vector Ward identity reads

$$\Gamma_4(p,0) = -\frac{\partial S^{-1}(p)}{\partial(-ip_4)}. \quad (8)$$

When $\mu \neq 0$, the above identity should still hold due to the fact that the corresponding $U(1)$ symmetry is not ruined by introducing μ. Therefore one has

$$\Gamma_4(p,0)[\mu] = -\frac{\partial S^{-1}(p)[\mu]}{\partial(-ip_4)}. \quad (9)$$

Substituting Eq. (9) into Eq. (7) one would obtain

$$\chi = -N_f N_c Z_2(\zeta^2, \Lambda^2) \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\Big\{\frac{\partial S(p)[\mu]}{\partial(-ip_4)}\gamma_4\Big\}. \quad (10)$$

On the other hand, from Eqs. (2) and (5) we can obtain

$$\chi = \frac{\partial\rho}{\partial\mu} = -N_f N_c Z_2(\zeta^2, \Lambda^2) \cdot \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\Big\{\frac{\partial S(p)[\mu]}{\partial\mu}\gamma_4\Big\}. \quad (11)$$

Comparing Eq. (9) and Eq. (10) one will find

$$\frac{\partial S(p)[\mu]}{\partial(-ip_4)} = \frac{\partial S(p)[\mu]}{\partial\mu}. \quad (12)$$

We find that the quark propagator is dependent on $(-ip_4)$ and μ in the same way, thus when p_4 appears at finite μ, it must have the form $p_4 + i\mu$, and vice versa. Furthermore, it has been proved in the framework of the rainbow-ladder approximation of the Dyson–Schwinger equations (DSEs) that under the approximation of ignoring the μ-dependence of the dressed gluon propagator (this is a commonly used approximation in calculating the dressed quark propagator at finite chemical potential[26,18]) and under the assumption that the dressed quark propagator at finite μ is analytic in the neighborhood of $\mu = 0$, the dressed quark propagator at finite μ can be obtained from the one at $\mu = 0$ by a simple shift[27]

$$S(q)[\mu] = S(\tilde{q}), \quad (13)$$

where $\tilde{q} = (\vec{q}, q_4 + i\mu)$. It should be noted that a similar relation is used widely in the perturbative thermal field theory.[28,29] From the above discussion, one can find that, even in QCD which is prominently non-perturbative, the symmetry guarantees the relation still holding. The proof can be easily extended to the finite temperature case. Furthermore one can use another quark-meson vertex and corresponding susceptibility to prove the relation as far as the similar Ward identity holds (for example, when ignoring the current quark mass one can use the axial-vector quark–meson vertex and the corresponding susceptibility).

With Eq. (12) one can find

$$\rho(\mu) = -N_c N_f Z_2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{S(\tilde{q})\gamma_4\}, \quad (14)$$

$$-\langle\bar{q}q\rangle[\mu] = N_c N_f Z_4 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{S(\tilde{q})\}. \quad (15)$$

Making use of the identity

$$\int_{C_0} dq_4 f(q_4 + i\mu) = \int_{C_1} dq_4 f(q_4), \quad (16)$$

where C_0 and C_1 are the two integration paths shown in Fig. 1. One can rewrite Eqs. (14) and (15) as

$$\rho(\mu) = -N_c N_f Z_2 \int_{C_1} \frac{dq_4}{2\pi} F_1(q_4), \tag{17}$$

$$-\langle\bar{q}q\rangle[\mu] = N_c N_f Z_4 \int_{C_1} \frac{dq_4}{2\pi} F_2(q_4), \tag{18}$$

with

$$F_1(q_4) \equiv \int \frac{d^3\vec{q}}{(2\pi)^3} \mathrm{tr}\{S(q)\gamma_4\}, \tag{19}$$

$$F_2(q_4) \equiv \int \frac{d^3\vec{q}}{(2\pi)^3} \mathrm{tr}\{S(q)\}. \tag{20}$$

Obviously when $\mu \to 0$ the integral path $C_1 \to C_0$ and one would obtain the quark number density and the quark condensate at $\mu = 0$. If we denote those poles of $F_1(q_4)$ and $F_2(q_4)$, they will locate in the upper half complex q_4 plane as

$$z_m^{F_1} = \chi_m^{F_1} + i\omega_m^{F_1}, \tag{21}$$

$$z_n^{F_2} = \chi_n^{F_2} + i\omega_n^{F_2}. \tag{22}$$

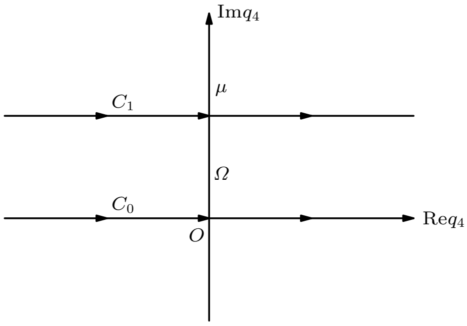

Fig. 1. The integration paths in the complex q_4 plane.

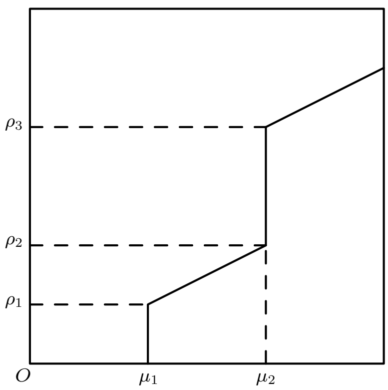

Fig. 2. Schematic dependence of the quark number density on the chemical potential at $T = 0$.

Then according to Cauchy's theorem we obtain

$$\rho(\mu) = \rho(\mu = 0) + iN_c N_f Z_2 \sum_m \theta(\mu - \omega_m^{F_1}) \mathrm{Res}\{F_1(z_m^{F_1})\}, \tag{23}$$

$$-\langle\bar{q}q\rangle[\mu] = -\langle\bar{q}q\rangle[\mu = 0] + iN_c N_f Z_4 \sum_n \theta(\mu - \omega_n^{F_2}) \mathrm{Res}\{F_2(z_n^{F_2})\}. \tag{24}$$

To obtain Eqs. (23) and (24), the vanishing integration on the infinite contour is used by the property of asymptotic freedom of quark propagator.

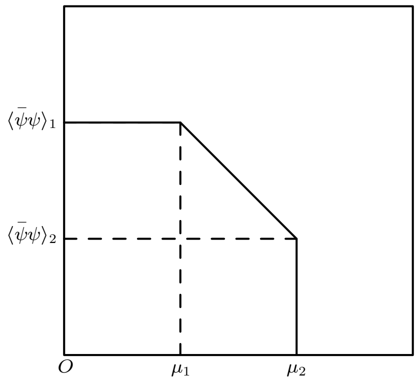

Fig. 3. Schematic dependence of the quark condensate on the chemical potential at $T = 0$.

From Eqs. (22) and (23) one would find that when $\mu < \min\{\omega_n^{F_1}\}$ (or $\mu < \min\{\omega_m^{F_2}\}$), the quark number density (or the quark condensate) will remain unchanged from its value at $\mu = 0$. Therefore some critical value of μ, below which the quark number density or the quark condensate is the same as the one at $\mu = 0$, must exist. Here it should be noted that in Ref. [30] based on a universal argument the authors found that the existence of some singularity of the quark number density at the point $\mu = \mu_0$ and $T = 0$ is a robust and model-independent prediction. Obviously Eq. (22) coincides with this conclusion.

Let us use μ_1 and μ_2 to denote the change point of the quark number density and the quark condensate, respectively. From Eqs. (23) and (24) one can find that μ_1 and μ_2 are determined by the position of the poles of F_1 and F_2, respectively. From Eqs. (19) and (20) and the expression of the dressed quark propagator as follows:

$$S(q) = \frac{1}{i\not{q}A(q^2) + B(q^2)}, \tag{25}$$

one can find that the poles of F_1 and F_2 can be obtained by solving the same equation

$$q^2 A^2(q^2) + B^2(q^2) = 0. \tag{26}$$

It is worth noting that it is unnecessary to require $A(q^2) = 0$ and $B(q^2) = 0$, respectively. Therefore the analytic structures of F_1 and F_2 in the complex q_4 plane should be the same. Thus we obtain the following result

$$\mu_1 = \mu_2. \tag{27}$$

Furthermore, it can be demonstrated that the quark number density is 0 at $T = 0$ and $\mu = 0$. According to Eq. (25), the quark propagator reads

$$S(q) = \frac{-i\not{q}A(q^2)}{q^2A^2(q^2) + B^2(q^2)} + \frac{B(q^2)}{q^2A^2(q^2) + B^2(q^2)}. \tag{28}$$

Substituting Eq. (28) into Eq. (14), one obtains

$$\rho(0) = -N_{\rm c}N_{\rm f}Z_2\int\frac{d^4q}{(2\pi)^4}4\frac{-iq_4A(q^2)}{q^2A^2(q^2)+B^2(q^2)}. \tag{29}$$

Since the integrand is an odd function independent of p_4, the quark number density is exactly 0 at $T=0$ and $\mu=0$. This implies that the quark number density is exactly 0 when $\mu<\mu_1$. Figure 2 is a schematic dependence of the quark number density on μ, the quark number density becomes nonzero at $\mu=\mu_1$ (it may be continuous). Similar to the quark number density, the quark condensate preserves $-\langle\bar\psi\psi\rangle[\mu=0]$ in the region of $\mu<\mu_1$. Figure 3 is a schematic dependence of the quark condensate on μ. The quark condensate is a nonzero value for $\mu<\mu_{\rm c}$, and it becomes zero at $\mu=\mu_{\rm c}$. This implies that the chiral phase transition happens at $\mu=\mu_{\rm c}$, as shown in Fig. 3. It would be the first-order phase transition if the quark condensate undergoes a jump from nonzero to zero, otherwise it would be the second-order phase transition.

In summary, we have studied the quark number density and the quark condensate of QCD at zero temperature and finite chemical potential. Both quark number density and quark condensate are dependent on the full quark propagator. In the chiral limit case, the quark condensate is a well-defined order parameter of the chiral phase transition. At zero temperature, $p_4\to p_4+i\mu$ is usually used to study dense QCD in rainbow truncation within the framework of DSEs. By means of the partition function of QCD and functional integral techniques, we give a general relation between the first-order derivative to $(-ip_4)$ and μ of quark propagator at finite μ and $T=0$. Using Cauchy's theorem, we discuss the quark number density and quark condensate by analyzing their singularities in complex plane of p_4. At zero temperature, we find that the positions at which the quark number density becomes nonzero and the quark condensate begins to change are located at the same μ. At some certain chemical potentials, the quark condensate becomes 0, in which the chiral phase transition happens.

References

[1] Xu S S, Yan Y, Cui Z F and Zong H S 2015 arXiv:1506.06846 [hep-ph]
[2] Takahashi J, Sugano J, Ishii M, Kouno H and Yahiro M 2014 arXiv:1410.8279
[3] Nagata K 2012 arXiv:1204.6480
[4] Haque N, Mustafa M G and Strickland M 2013 *J. High Energy Phys.* 1307 184
[5] Takahashi J, Kouno H and Yahiro M 2015 *Phys. Rev.* D **91** 014501
[6] Andersen J O, Mogliacci S, Su N and Vuorinen A 2013 *Phys. Rev.* D **87** 074003
[7] Miura K, Lombardo M P and Pallante E 2012 *Phys. Lett.* B **710** 676
[8] Bhattacharya T, Buchoff M I, Christ N H, Ding H T, Gupta R, Jung C, Karsch F, Lin Z, Mawhinney R D, McGlynn G, Mukherjee S, Murphy D, Petreczky P, Renfrew D, Schroeder C, Soltz R A, Vranas P M and Yin H 2014 *Phys. Rev. Lett.* **113** 082001
[9] Kaczmarek O, Karsch F, Laermann E, Miao C, Mukherjee S, Petreczky P, Schmidt C, Soeldner W and Unger W 2011 *Phys. Rev.* D **83** 014504
[10] Yin P l, Cui Z F, Feng H T and Zong H S 2014 *Ann. Phys.* **348** 306
[11] Bazavov A, Bhattacharya T, Cheng M, DeTar C, Ding H T, Gottlieb S, Gupta R, Hegde P, Heller U M, Karsch F, Laermann E, Levkova L, Mukherjee S, Petreczky P, Schmidt C, Soltz R A, Soeldner W, Sugar R, Toussaint D, Unger W and Vranas P 2012 *Phys. Rev.* D **85** 054503
[12] Cui Z F, Shi C, Xia Y H, Jiang Y and Zong H S 2013 *Eur. Phys. J.* C **73** 2612
[13] Morais J, Moreira J, Hiller B, Blin A and Osipov A 2014 arXiv:1411.3203
[14] Xu J, Song T, Ko C M and Li F 2014 *Phys. Rev. Lett.* **112** 012301
[15] Zhu H X, Sun W M and Zong H S 2013 *Chin. Phys. Lett.* **30** 051201
[16] Xin X Y, Qin S X and Liu Y X 2014 *Phys. Rev.* D **90** 076006
[17] Wang B, Wang Y L, Cui Z F and Zong H S 2015 *Phys. Rev.* D **91** 034017
[18] Xu S S, Cui Z F, Wang B, Shi Y M, Yang Y C and Zong H S 2015 *Phys. Rev.* D **91** 056003
[19] Jiang Y, Hou F Y, Luo C B and Zong H S 2015 *Chin. Phys. Lett.* **32** 021201
[20] Tian Y L, Cui Z F, Wang B, Shi Y M, Yang Y C and Zong H S 2015 *Chin. Phys. Lett.* **32** 081101
[21] Maris P, Roberts C D and Tandy P C 1998 *Phys. Lett.* B **420** 267
[22] Gavai R V, Potvin J and Sanielevici S 1989 *Phys. Rev.* D **40** 2743
[23] Gottlieb S, Liu W, Renken R L, Sugar R L and Toussaint D 1988 *Phys. Rev.* D **38** 2888
[24] He M, Li J F, Sun W M and Zong H S 2009 *Phys. Rev.* D **79** 036001
[25] Cui Z F, Hou F Y, Shi Y M, Wang Y L and Zong H S 2015 *Ann. Phys. (N. Y.)* **358** 172
[26] Qin S X, Chang L, Chen H, Liu Y X and Roberts C D 2011 *Phys. Rev. Lett.* **106** 172301
[27] Zong H S and Sun W M 2008 *Phys. Rev.* D **78** 054001
[28] Fraga E S and Romatschke P 2005 *Phys. Rev.* D **71** 105014
[29] Andersen J O and Strickland M 2002 *Phys. Rev.* D **66** 105001
[30] Halasz M A, Jackson A D, Shrock R E, Stephanov M A and Verbaarschot J J M 1998 *Phys. Rev.* D **58** 096007