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RESEARCH ARTICLE

# Quintessence and phantom emerging from the split-complex field and the split-quaternion field

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**Abstract** Motivated by the mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers) and the split-quaternion numbers (or coquaternion numbers), we define the notion of split-complex scalar field and the split-quaternion scalar field. Then we explore the cosmic evolution of these scalar fields in the background of spatially flat Friedmann–Robertson–Walker Universe. We find that both the quintessence field and the phantom field could naturally emerge in these scalar fields. Introducing the metric of field space, these theories fall into a subclass of the multi-field theories which have been extensively studied in inflationary cosmology.

**Keywords** Split-complex field · Split-quaternion field · Quintessence · Phantom

## 1 Introduction

More and more accurate and convincing astronomical observations [1–3] indicate that dark energy dominates our current Universe. Although a host of observationally viable

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dark energy models have been proposed, the nature of dark energy is still undetermined. The Einstein cosmological constant is in many respects the most economical solution to the problem of dark energy. But it is confronted with two fundamental problems: the fine tuning problem and the coincidence problem (see e.g. [4]). So one turn to the study of other options for dark energy, for example, quintessence [5–10], quintom [11–17], k-essence [18–20], Chaplygin gas [21,22], holographic dark energy [23–43] and so on.

One find that in many models of quintessence field, there exist the so-called tracker solutions. In these solutions, the quintessence field always has a energy density closely tracks (but is smaller than) that of the radiation until the matter-radiation equality. By this way, the coincidence problem is solved [5–10]. Phantom energy is introduced into the study of cosmic evolution by Caldwell [44,45]. The Lagrangian of phantom takes the form $\mathscr{L}_\phi = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)$, with $\phi$ the phantom field and $V(\phi)$ the phantom potential. By changing the sign of kinetic term in quintessence field *by hand,* this form of Lagrangian could be obtained. One then find that the energy density of phantom actually increases with cosmic time. So the fate of the Universe is a Big Rip.

Now we want to ask: can we unify quintessence and phantom in a natural way? The answer is yes. We find that the quintessence and phantom can naturally emerge in the theory of split-complex scalar field, the split-quaternion scalar field and the split-octonion scalar field. This finding is motivated by the mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers), the split-quaternion numbers (or coquaternion numbers) and the split-octonion numbers [46] (for references, see also the Wikipedia[1]).

We note that our application of split-complex number is not just a mathematical tool but rather it has physical information too. Actually, the theory of split-complex number has been applied in gravitational fields [47], quantum group [48], quantum mechanics [49], quintessence cosmology [50,51] and so on. What is more, with the help of split-complex variables, a pseudo-complex field theory [52] and a pseudo-complex general relativity [53] are recently presented. For the split-complex scalar field, if one require it obeys the symmetry of invariance under hyperbolic rotation, the split-complex field would restore to the Hessence field [54,55]. On the other hand, if the symmetry is allowed to be broken, the split-complex scalar field would turn out to be the quintom field [11–17]. Coleman found that for conventional complex field, there exists Q-Balls solutions [56] due to the conserved charge. We find there is also conserved charge for split-complex field. So we expect Q-Balls-like solutions exist which makes the split-complex field more physical. Finally, we find the split-complex scalar field could be re-formulated as one specific model of the multi-filed theory which attracts a lot of effort in the study of inflationary universe [57–61] in recent years. In all, the split scalar fields have rich physics.

The paper is organized as follows. In Sect. 2, we define the notion of split-complex scalar field and show quintessence and phantom could emerge from this field. In Sect. 3, we investigate the cosmic evolution of the split-complex field and show that the detail of dynamics is closely related to the initial conditions on the quintessence and phantom. In Sect. 4, we investigate the cosmic evolution of the split-quaternion field.

1 http://en.wikipedia.org/wiki/Split-complex_number.

In Sect. 5, the linear perturbations of these fields in the background of Friedmann-Robertson-Walker Universe are present. Conclusions and discussions are given in Sect. 6. Throughout this paper, we adopt the system of units in which $G = c = \hbar = 1$ and the metric signature $(-, +, +, +)$.

## 2 Split-complex scalar field

### 2.1 What is split-complex scalar field

The mathematic theory of split-complex numbers (or hyperbolic numbers, also perplex numbers) can be found in Ref. [46] or the Wikipedia[2]. Motivated by the theory of these numbers, we define the split-complex scalar field $\Phi$ as follows

$$\Phi = \phi + j\psi, \tag{1}$$

where $\phi$ and $\psi$ are two real scalar fields. The quantity $j$ is similar to the imaginary unit $i$ except that [46]

$$j^2 = +1. \tag{2}$$

Choosing $j^2 = -1$ results in the conventional complex scalar field. It is this change of sign which distinguishes the split-complex scalar field from the ordinary complex one. The quantity $j$ here is not a real number but an independent quantity. Namely, it is not equal to $\pm 1$.

Just as for ordinary complex field, one can define the notion of a split-complex conjugate as follows

$$\Phi^* = \phi - j\psi. \tag{3}$$

Then the modulus of a split-complex scalar field is given by the isotropic quadratic form

$$\Phi\Phi^* = \phi^2 - \psi^2. \tag{4}$$

This quadratic form is split into positive and negative parts, in contrast to the positive definite form of the ordinary complex scalar field.

Similar to the ordinary complex field which can be written in the form of Euler's formula

$$\Phi = \phi e^{i\theta} = \phi\cos\theta + i\phi\sin\theta, \tag{5}$$

the split-complex field has the Euler's formula as follows

$$\Phi = \phi e^{j\theta} = \phi\cosh\theta + j\phi\sinh\theta. \tag{6}$$

[2] http://en.wikipedia.org/wiki/Split-complex_number.

This can be derived from a power series expansion using the fact that *cosh* has only even powers while that for *sinh* has odd powers. It follows that $\Phi\Phi^* = \phi^2$.

### 2.2 Quintessence and phantom from split-complex field

We shall consider the theory of a massive, split-complex, self-interacting scalar field with the Lagrangian density as follows

$$\mathscr{L} = \frac{1}{2}\nabla_\mu \Phi \nabla^\mu \Phi^* + \frac{1}{4}\lambda^2 \left(\Phi\Phi^* + m^2/\lambda^2\right)^2, \tag{7}$$

with $m$ the mass of the scalar and $\lambda$ a coupling constant. Here the potential is the same as minimally coupled Higgs potential which has been given experimental evidence. Of course, one may consider other potentials, i.e, exponential potential, power-law potential and so on. However, if one demand the theory obeys the symmetry of hyperbolic rotation (see subsection C below), the potential should be constrained to be the function of $\Phi\Phi^*$, namely $V(\Phi\Phi^*)$. On the other hand, if the symmetry is broken, the potential could be $V(\Phi + \Phi^*,\ \Phi - \Phi^*)$. In the next, we will see the former is exactly the Hessence field [54,55] and the latter the quintom field [11–17]. So the split-complex scalar field procedure is not just a mathematical re-formulation of the Hessence or Quintom, but has the physical meaning of internal symmetry.

Substituting Eq. (1) into Eq. (7), we obtain

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi \\ &+ \frac{1}{4}\lambda^2 \left(\phi^2 - \psi^2 + m^2/\lambda^2\right)^2.\end{aligned} \tag{8}$$

If we take the field $\Phi$ in the Lagrangian Eq. (7) as the ordinary complex scalar field, $\Phi = \phi + i\psi$, the Lagrangian takes the form

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi + \frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi \\ &+ \frac{1}{4}\lambda^2 \left(\phi^2 + \psi^2 + m^2/\lambda^2\right)^2.\end{aligned} \tag{9}$$

It is apparent there is sign of difference before the terms $\frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi$ and $\psi^2$. This is due to the fact that $i^2 = -1$ and $j^2 = 1$. One can recognize that Lagrangian, Eq. (8) is nothing but the Hessence field proposed by Wei et al in Ref. [54,55]. $\phi$ and $\psi$ plays the role of quintessence and phantom, respectively. The difference of Hessence field from the quintom field [11–17]

$$\mathscr{L} = \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi + V(\phi, \psi), \tag{10}$$

is that the form of the scalar potential is greatly constrained by $V(\phi^2 - \psi^2)$ in Hessence. We point out that there is a remarkable difference in our motivation from the Hessence. Ref. [54,55] propose the Lagrangian density of Hessence as follows

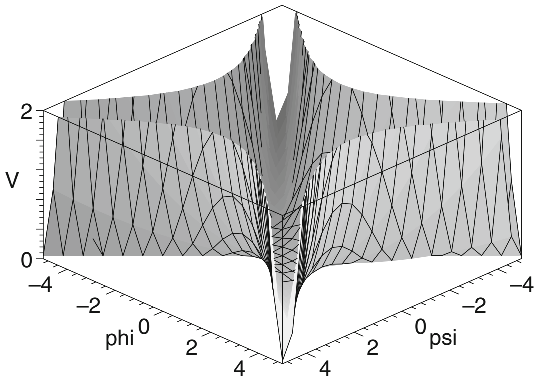


**Fig. 1** The sketch of the potential $V \propto (\Phi\Phi^* + M^2)^2$ with the quintessence field $\phi$ and the phantom field $\psi$. The potential has the vanishing absolute vacuum energy on the hyperbola $\phi^2 - \psi^2 + M^2 = 0$

$$\mathscr{L} = \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi + \frac{1}{2}\nabla_\mu\Phi^*\nabla^\mu\Phi^* + V\left(\Phi^2 + \Phi^{*2}\right), \tag{11}$$

with $\Phi$ the conventional complex scalar field. However, our Lagrangian density Eq. (7) is different from Eq. (11) not only on the Lagrangian expression but also on the physical meaning of scalar field $\Phi$. In Fig. 1 we plot the scalar potential $V \propto (\Phi\Phi^* + M^2)^2$ with the quintessence field $\phi$ and the phantom field $\psi$. The potential has the vanishing absolute vacuum energy on the hyperbola $\phi^2 - \psi^2 + M^2 = 0$.

### 2.3 Symmetry

The theory of Lagrangian Eq. (7) has the symmetry that it is invariant after a hyperbolic rotation

$$\Phi \to \Phi e^{j\alpha}, \tag{12}$$

with $\alpha$ a constant. Expressed the split-complex scalar field in terms of two real fields $\phi$ and $\psi$, the hyperbolic rotation corresponds to the $O(1,1)$ transformations

$$\phi \to \phi\cosh\alpha + \psi\sinh\alpha, \tag{13}$$
$$\psi \to \phi\sinh\alpha + \psi\cosh\alpha. \tag{14}$$

Then the Noë̈ther theorem tells us this symmetry leads to a conserved charge which is given by the formula

$$Q = \frac{1}{2j}\int d^3x\left(\Phi^*\partial_0\Phi - \Phi\partial_0\Phi^*\right), \tag{15}$$

in the background of four dimensional Minkowski spacetime. Here $\partial_0$ represents the derivative with the time. Coleman found that for conventional complex field, there would exist Q-Balls solutions [56] due to the conserved charge. Since there is also conserved charge for split-complex field, we expect Q-Balls-like solutions also exist which makes the split-complex field more physical.

### 2.4 The stability

Introduce the metric tensor $\eta_{IJ}$ in the field space $\Phi_I = (\phi, \psi)$,

$$\eta_{IJ} = \eta^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

where $I, J = 1, 2$. Eq. (8) can be expressed as

$$\mathscr{L} = \frac{1}{2}\nabla_\mu \Phi_I \nabla^\mu \Phi^I + \frac{1}{4}\lambda^2 \left(\Phi_I \Phi^I + m^2/\lambda^2\right)^2. \tag{17}$$

With this form, the theory belongs to the general multiple scalar field theory considered in RefS. [57–61]. The equation of motion and the energy momentum are give by

$$\nabla_\mu \nabla^\mu \Phi_I - \lambda^2 \left(\Phi_J \Phi^J + m^2/\lambda^2\right) \Phi_I = 0, \tag{18}$$

and

$$T_{\mu\nu} = -\nabla_\mu \Phi_I \nabla^\mu \Phi^I + g_{\mu\nu}\mathscr{L}. \tag{19}$$

From the equation of motion, we see both the quintessence $\phi$ and the phantom $\psi$ acquire a vanishing effective mass $m_0$:

$$m_0 = \lambda\sqrt{\Phi_J \Phi^J + m^2/\lambda^2}|_{(\Phi_J \Phi^J + m^2/\lambda^2)} = 0, \tag{20}$$

at the global minimum of the potential. In this case, not only quintessence but also phantom is stable in dynamics.

This is different from the usual phantom field with the Lagrangian

$$\mathscr{L} = -\frac{1}{2}\nabla_\mu \psi \nabla^\mu \psi + V(\psi), \tag{21}$$

and the equation of motion

$$\nabla_\mu \nabla^\mu \psi + m_0^2 \psi = 0, \tag{22}$$

where $m_0^2 \equiv V_{,\psi\psi}\ |_{\psi=\psi_0}$. The value of $\psi = \psi_0$ corresponds to the global minimum of the potential. In this case, the phantom field acquires an *imaginary* mass which would lead to the classically and quantum instability. Then why is there such a difference in the mass? The reason are as follows. The usual phantom scalar potential has the form $V = \frac{1}{2}m_0^2\psi^2$ at the global minimum. However, for the Lagrangian Eq. (8), the phantom potential takes the form $V = -\frac{1}{2}m_0^2\psi^2$ at the global minimum. Thus it is very important that the potential of the split-complex scalar is constrained to be $V(\phi^2 - \psi^2)$ in order to avoid the problem of instability.

### 2.5 Conformal invariant split-complex field

Recently, Kallosh and Linde [62] proposed a simple conformally invariant two-field model of dS/AdS space. The model consists of two real scalar fields, $\phi$ and $\psi$, which has the $SO(1,1)$ symmetry:

$$\mathscr{L}_{KL} = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\nabla_\mu\psi\nabla^\mu\psi + \frac{1}{12}\left(\phi^2 - \psi^2\right)R \\ -\frac{1}{4}\lambda(\phi^2 - \psi^2)^2. \tag{23}$$

Here $R$ is the Ricci scalar and $\lambda$ a coupling constant. This theory is locally conformal invariant under the following transformations,

$$\tilde{g}_{\mu\nu} = e^{-2\sigma(x)}g_{\mu\nu}, \quad \tilde{\phi} = e^{\sigma(x)}\phi, \quad \tilde{\psi} = e^{\sigma(x)}\psi. \tag{24}$$

The global $SO(1,1)$ symmetry is a boost between these two fields. Using the concept of our split-complex scalar field $\Phi_I$, the theory is equivalent to

$$\mathscr{L} = \frac{1}{2}\nabla_\mu\Phi_I\nabla^\mu\Phi^I + \frac{1}{12}R\Phi_I\Phi^I - \frac{1}{4}\lambda\left(\Phi_I\Phi^I\right)^2. \tag{25}$$

In other words, the two scalar fields considered in Ref. [62] is exactly the conformal invariant split-complex field.

## 3 Dynamics of split-complex scalar field

In this section, we investigate the cosmic evolution of the split-complex field in the background of spatially flat Friedmann–Robertson–Walker (FRW) Universe

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2), \tag{26}$$

where $a(t)$ is the cosmic scale factor. We model all other matter sources present in the Universe as perfect fluids. These matter sources can be baryonic matter, relativistic matter and dark energy. We assume there is no interaction between the split-complex scalar field and other matter fields, other than by gravity. Then the Einstein equations and the equation of motion of the scalar fields are given by

$$3H^2 = \kappa^2\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 + V + \rho_r + \rho_m\right), \tag{27}$$

$$2\dot{H} + 3H^2 = -\kappa^2\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 - V + \frac{1}{3}\rho_r\right), \tag{28}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{29}$$
$$\ddot{\psi} + 3H\dot{\psi} - V_{,\psi} = 0, \tag{30}$$

respectively. Here $H \equiv \dot{a}/a$ denotes the Hubble parameter and dot is the derivative with respect to cosmic time, $t$. $\rho_m$ and $\rho_r$ are the energy density of dark matter and relativistic matter. $V_{,\phi}$ and $V_{,\psi}$ denote the derivative with respect to $\phi$ and $\psi$, respectively. Introduce the following dimensionless quantities

$$\begin{aligned}
&X \equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}, \quad Y \equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\psi}}{H}, \\
&Z \equiv \frac{\kappa}{\sqrt{6}}\frac{\sqrt{V}}{H}, \quad U \equiv \frac{\kappa}{\sqrt{3}}\frac{\sqrt{\rho_m}}{H}, \\
&\lambda_\phi \equiv -\frac{2}{\sqrt{6}}\frac{V_{,\phi}}{\kappa V}, \quad \lambda_\psi \equiv -\frac{2}{\sqrt{6}}\frac{V_{,\psi}}{\kappa V}, \\
&N \equiv \ln a,
\end{aligned} \tag{31}$$

then the equations of motion can be written in the following autonomous form

$$\begin{aligned}
\frac{dX}{dN} &= -3X - \frac{3}{2}\lambda_\phi Z^2 - X\frac{\dot{H}}{H^2}, \\
\frac{dY}{dN} &= -3Y - \frac{3}{2}\lambda_\psi Z^2 - Y\frac{\dot{H}}{H^2}, \\
\frac{dZ}{dN} &= -\frac{3}{2}Z\left(\lambda_\phi X + \lambda_\psi Y\right) - Z\frac{\dot{H}}{H^2}, \\
\frac{dU}{dN} &= -\frac{3}{2}U - U\frac{\dot{H}}{H^2}, \\
\frac{d\lambda_\phi}{dN} &= \frac{3X\left(\lambda_\psi^2 - \lambda_\phi^2\right)}{2 + 2\sqrt{1 - s^2\left(\lambda_\phi^2 - \lambda_\psi^2\right)}} \\
&\quad + \frac{3}{2}\lambda_\phi\left(X\lambda_\phi - Y\lambda_\psi\right), \\
\frac{d\lambda_\psi}{dN} &= \frac{3Y\left(\lambda_\phi^2 - \lambda_\psi^2\right)}{2 + 2\sqrt{1 - s^2\left(\lambda_\phi^2 - \lambda_\psi^2\right)}} \\
&\quad + \frac{3}{2}\lambda_\psi\left(X\lambda_\phi - Y\lambda_\psi\right),
\end{aligned} \tag{32}$$

together with a constraint equation

$$X^2 - Y^2 + Z^2 + U^2 + \frac{\kappa^2 \rho_r}{3H^2} = 1. \tag{33}$$

Here

$$s = \frac{\sqrt{6}}{4}\frac{m\kappa}{\lambda}, \tag{34}$$

and

$$\frac{\dot{H}}{H^2} = -2 - X^2 + Y^2 + 2Z^2 + \frac{1}{2}U^2. \tag{35}$$

The equation of state $w$ and the fraction of the energy density for the split-complex scalar field are

$$w \equiv \frac{X^2 - Y^2 - Z^2}{X^2 - Y^2 + Z^2},$$
$$\Omega_\Phi \equiv X^2 - Y^2 + Z^2, \tag{36}$$

As an example, we consider $s = 1$. Physically, the quintessence $\phi$ would roll down the potential and the phantom $\psi$ roll up the potential. Therefore we can safely assume $X > 0, Y > 0, \lambda_\phi > 0, \lambda_\psi < 0$. We investigate the cosmology model with the following values of parameters: $\Omega_{k0} = 0, \Omega_{m0} = 0.27, \Omega_{r0} = 8.1 \cdot 10^{-5}, \Omega_{X0} = 0.73$, which are consistent with current observations [63,64].

In Fig. 2, we plot the phase plane for the evolution of the split-complex scalar with a range of different initial conditions. The point (0, 0, 0) corresponds to the radiation or matter dominated epoches. The point (0, 0, 0) is unstable and the point (0, 0, 1) is stable and thus an attractor. These trajectories show that the Universe always ends at the split-complex scalar potential dominated epoch.

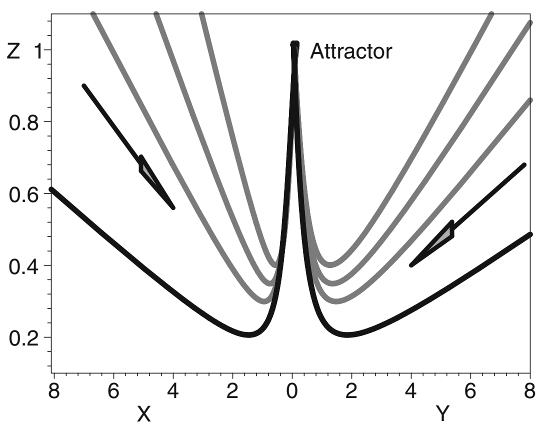


**Fig. 2** The phase plane for the evolution of the split-complex scalar with a range of different initial conditions. The point (0, 0, 0) corresponds to the radiation or matter dominated epoches. The point (0, 0, 0) is unstable and the point (0, 0, 1) is stable and thus an attractor. These trajectories show that the Universe always ends at the split-complex scalar potential dominated epoch

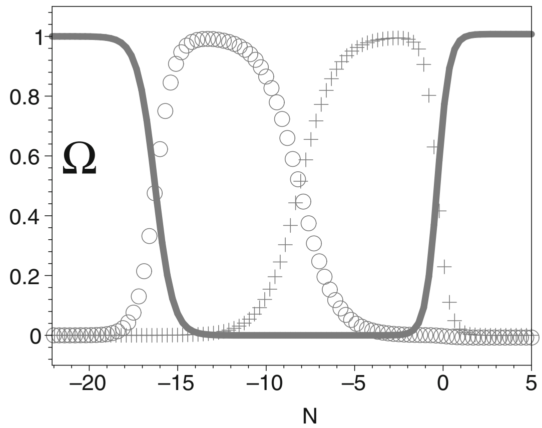


**Fig. 3** The evolution of density fractions for radiation (*circled line*), dark matter (*crossed line*) and the split-complex scalar field (*solid line*)

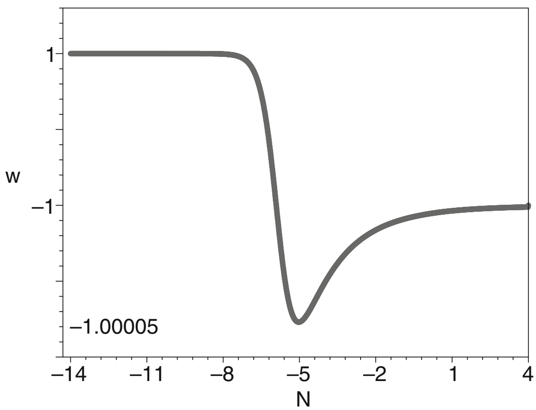


**Fig. 4** The evolution of the equation of state for the split-complex scalar. It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts. The split-complex scalar can cross the phantom divide around the value of $N = -5$ (redshift 3000)

In Fig. 3, we plot the evolution of density fractions for radiation, dark matter and the split-complex scalar field. This shows that the split-complex scalar field can mimic the dark energy very well.

In Fig. 4, we plot the evolution of the equation of state for the split-complex scalar. It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts. It can cross the phantom divide around the value of $N = -5$ (redshift 3000).

## 4 Dynamics of split-quaternion scalar field

### 4.1 Split-quaternion scalar field

In this subsection, we define the notion of split-quaternion scalar field and the split-octonion field. We find they actually consists of two quintessence fields and two phantom fields. The number of quintessence is exactly the same as that of the phantom. To this end, we start from the theory of split-quaternion number $q$ (or coquaternion) which is given by [65][3]

$$q = u + ix + jy + kz, \tag{37}$$

where $u, x, y, z$ are real numbers. The quantity $i, j, k$ here are not real numbers but independent quantities. The products of these elements are [65]

$$\begin{aligned} ij &= k = -ji, \\ jk &= -i = -kj, \\ ki &= j = -ik, \\ i^2 &= -1, \\ j^2 &= +1, \\ k^2 &= +1, \end{aligned} \tag{38}$$

and hence $ijk = 1$. A split-quaternion has a conjugate

$$q^* = u - ix - jy - kz, \tag{39}$$

and multiplicative modulus

$$qq* = u^2 + x^2 - y^2 - z^2. \tag{40}$$

This quadratic form is split into positive and negative parts, in contrast to the positive definite form on the algebra of quaternions.

We now define the split-quaternion scalar field $\Phi$ as

$$\Phi = \phi_1 + i\phi_2 + j\psi_1 + k\psi_2. \tag{41}$$

Then the Lagrangian

$$\mathscr{L} = \frac{1}{2}\nabla_\mu \Phi \nabla^\mu \Phi^* + \frac{1}{4}\lambda^2 \left(\Phi\Phi^* + \frac{6\Lambda}{\kappa^2}\right)^2, \tag{42}$$

can be written as

$$\begin{aligned} \mathscr{L} = &\frac{1}{2}\nabla_\mu \phi_1 \nabla^\mu \phi_1 + \frac{1}{2}\nabla_\mu \phi_2 \nabla^\mu \phi_2 \\ &-\frac{1}{2}\nabla_\mu \psi_1 \nabla^\mu \psi_1 - \frac{1}{2}\nabla_\mu \psi_2 \nabla^\mu \psi_2 \\ &+\frac{1}{4}\lambda^2 \left(\phi_1^2 + \phi_2^2 - \psi_1^2 - \psi_2^2 + \frac{6\Lambda}{\kappa^2}\right)^2. \end{aligned} \tag{43}$$

[3] http://en.wikipedia.org/wiki/Split-quaternion.

Here $\Lambda$ is a positive constant. It is apparent the theory consists of two quintessence field $\phi_1, \phi_2$ and two phantom field $\psi_1, \psi_2$. Introduce the metric tensor $\eta_{IJ}$ in the space of field $\Phi_I = (\phi_1, \phi_2, \psi_1, \psi_2)$,

$$\eta_{IJ} = \eta^{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{44}$$

where $I, J = 1, 2, 3, 4$. Eq. (43) can be expressed as

$$\mathscr{L} = \frac{1}{2}\nabla_\mu \Phi_I \nabla^\mu \Phi^I + \frac{1}{4}\lambda^2 \left(\Phi_I \Phi^I + \frac{6\Lambda}{\kappa^2}\right)^2. \tag{45}$$

With this form, the theory also belongs to the general multiple scalar field theory considered in Refs. [57–61]. Furthermore, we could define the split-octonion field (motivated by the split-octonion[4]) by introducing the metric tensor in the space of field $\Phi_I = (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \psi_2, \psi_3, \psi_4)$,

$$\eta_{IJ} = \eta^{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{46}$$

where $I, J = 1, 2, 3, 4, 5, 6, 7, 8$. The Lagrangian Eq. (45) can also describe a massive, self-interacting split-octonion scalar field.

### 4.2 Dynamics

In this section, we study the dynamics of the split-quaternion scalar field in the background of spatially flat FRW Universe. For simplicity, we only consider the cosmic evolution of the split-quaternion scalar field.

The Einstein equations and the equation of motion of the scalar fields take the form

$$\begin{aligned} 3H^2 &= \kappa^2 \left(\frac{1}{2}\dot{\Phi}_I \dot{\Phi}^I + V\right), \\ 2\dot{H} + 3H^2 &= -\kappa^2 \left(\frac{1}{2}\dot{\Phi}_I \dot{\Phi}^I - V\right), \end{aligned} \tag{47}$$

[4] http://en.wikipedia.org/wiki/Split-octonion.

and

$$\ddot{\Phi}_I + 3H\dot{\Phi}_I + V_{,\Phi^I} = 0, \tag{48}$$

respectively.

Introduce the following dimensionless quantities

$$\begin{aligned}
X &\equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}_1}{H}, \quad \lambda_{\phi_1} \equiv \frac{3}{\sqrt{6}\kappa}\frac{V_{,\phi_1}}{V},\\
Y &\equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}_2}{H}, \quad \lambda_{\phi_2} \equiv \frac{3}{\sqrt{6}\kappa}\frac{V_{,\phi_2}}{V},\\
Z &\equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\psi}_1}{H}, \quad \lambda_{\psi_1} \equiv -\frac{3}{\sqrt{6}\kappa}\frac{V_{,\psi_1}}{V},\\
U &\equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\psi}_2}{H}, \quad \lambda_{\psi_2} \equiv -\frac{3}{\sqrt{6}\kappa}\frac{V_{,\psi_2}}{V},\\
N &\equiv \ln a,
\end{aligned} \tag{49}$$

then the equations of motion can be written in the following autonomous form

$$\begin{aligned}
\frac{dX}{dN} &= -3X - XB - \lambda_{\phi_1}A,\\
\frac{dY}{dN} &= -3Y - YB - \lambda_{\phi_2}A,\\
\frac{dZ}{dN} &= -3Z - ZB - \lambda_{\psi_1}A,\\
\frac{dU}{dN} &= -3U - UB - \lambda_{\psi_2}A,\\
\frac{d\lambda_{\phi_1}}{dN} &= -X\lambda_{\phi_1}^2 - Y\lambda_{\phi_1}\lambda_{\phi_2} - Z\lambda_{\phi_1}\lambda_{\psi_1} - U\lambda_{\phi_1}\lambda_{\psi_2}\\
&\quad + \frac{XC}{1+\sqrt{1-\Lambda C}},\\
\frac{d\lambda_{\phi_2}}{dN} &= -X\lambda_{\phi_1}\lambda_{\phi_2} - Y\lambda_{\phi_2}^2 - Z\lambda_{\phi_2}\lambda_{\psi_1} - U\lambda_{\phi_2}\lambda_{\psi_2}\\
&\quad + \frac{YC}{1+\sqrt{1-\Lambda C}},\\
\frac{d\lambda_{\psi_1}}{dN} &= X\lambda_{\phi_1}\lambda_{\psi_1} + Y\lambda_{\phi_2}\lambda_{\psi_1} + Z\lambda_{\psi_1}^2 + U\lambda_{\psi_2}\lambda_{\psi_1}\\
&\quad + \frac{ZC}{1+\sqrt{1-\Lambda C}},\\
\frac{d\lambda_{\psi_2}}{dN} &= X\lambda_{\phi_1}\lambda_{\psi_2} + Y\lambda_{\phi_2}\lambda_{\psi_2} + Z\lambda_{\psi_1}\lambda_{\psi_2} + U\lambda_{\psi_2}^2\\
&\quad + \frac{UC}{1+\sqrt{1-\Lambda C}},
\end{aligned}$$

$$\begin{aligned} A &\equiv 1 - X^2 - Y^2 + Z^2 + U^2, \\ B &\equiv -3\left(X^2 + Y^2 - Z^2 - U^2\right), \\ C &\equiv \lambda_{\phi_1}^2 + \lambda_{\phi_2}^2 - \lambda_{\psi_1}^2 - \lambda_{\psi_2}^2, \end{aligned} \tag{50}$$

together with a constraint equation

$$X^2 + Y^2 - Z^2 - U^2 + \frac{\kappa^2 V}{3H^2} = 1. \tag{51}$$

The equation of state $w$ of the split-quaternion scalar field is

$$w \equiv -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 - \frac{2}{3}B. \tag{52}$$

We note that the coupling constant $\lambda$ is not present in above equations. So there is only one free parameter, $\Lambda$. As an example, we consider $\Lambda = 1$. Physically, the quintessence $\phi_1, \phi_2$ would roll down the potential and the phantom $\psi_1, \psi_2$ roll up the potential. Therefore we shall assume $\lambda_{\phi_1} > 0, \lambda_{\phi_2} > 0, \lambda_{\psi_1} < 0, \lambda_{\psi_2} < 0, X < 0, Y < 0, Z > 0, U > 0$.

In Figs. 5 and 6, we plot the evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1$, $\phi_2$ and the phantom fields $\psi_1$, $\psi_2$, respectively, with respect to the equation of state $w$. They show that if the kinetic energy of quintessence dominates over that of phantom initially, $X^2 + Y^2 > Z^2 + U^2$ $(X(0) = -1.22,\ Y(0) = -0.95,\ Z(0) = 0.99,\ U(0) = 0.70,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) =$

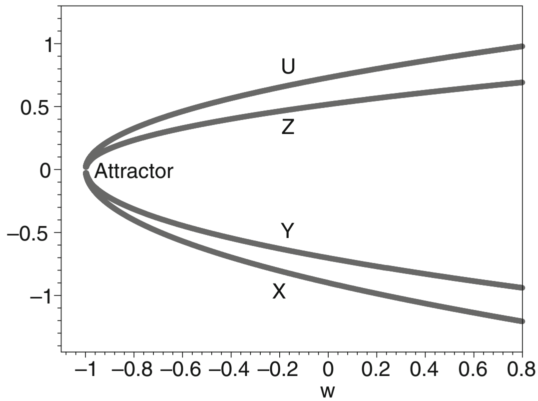


**Fig. 5** Evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1$, $\phi_2$ and the phantom fields $\psi_1$, $\psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -1.22,\ Y(0) = -0.95,\ Z(0) = 0.99,\ U(0) = 0.70,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) = -1/20,\ \lambda_{\psi_2}(0) = -1/20$. In other words, we set the same strength of fields but different scaled velocities. Here and in the afterwards, $(\cdots)(0) \equiv (\cdots)(N = 0)$. The figure shows that if the kinetic energy of quintessence dominates over that of phantom initially (with the same strength of fields), $X^2 + Y^2 > Z^2 + U^2$, the equation of state of the split-quaternion field would evolve from $+1$ to $-1$ with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor

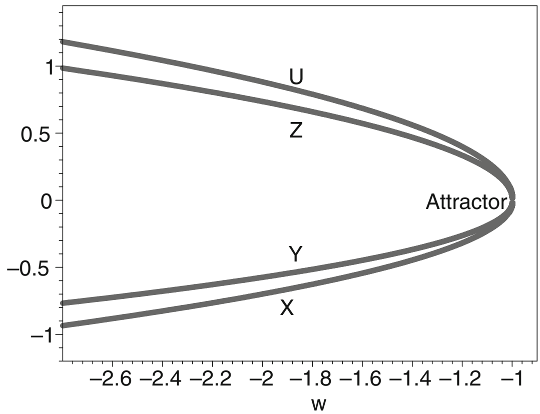


**Fig. 6** Evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1$, $\phi_2$, the phantom fields $\psi_1$, $\psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -0.95,\ Y(0) = -0.78,\ Z(0) = 1.2,\ U(0) = 1.0,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) = -1/20,\ \lambda_{\psi_2}(0) = -1/20$. In other words, we set the same strength of the fields but different scaled velocities. The figure shows that if the kinetic energy of phantom dominates over that of quintessence initially, $X^2 + Y^2 < Z^2 + U^2$ (with the same strength of fields), the equation of state of the split-quaternion field would evolve from minus infinity to $-1$ with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor

$-1/20,\ \lambda_{\psi_2}(0) = -1/20)$, the equation of state of the split-quaternion field would evolve from $+1$ to $-1$ with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor (Fig. 5). On the other hand, if the kinetic energy of phantom dominates over that of quintessence initially, $X^2 + Y^2 < Z^2 + U^2$ $(X(0) = -0.95,\ Y(0) = -0.78,\ Z(0) = 1.2,\ U(0) = 1.0,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) = -1/20,\ \lambda_{\psi_2}(0) = -1/20)$, the equation of state of the split-quaternion field would evolve from minus infinity to $-1$ with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor (Fig. 6). Corresponding to Figs. 5 and 6, we plot the equation of state in Figs. 7 and 8, respectively.

In Fig. 9, we plot the evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1, \phi_2$, the phantom fields $\psi_1, \psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -1,\ Y(0) = -1.1,\ Z(0) = 1,\ U(0) = 1.1,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) = -1/27,\ \lambda_{\psi_2}(0) = -1/20$. The figure shows that the equation of state of the split-quaternion field could cross the phantom divide with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor.

In Fig. 10, we plot evolution of the scaled velocities $X$, $Y$, $Z$, $U$ for the quintessence fields $\phi_1$, $\phi_2$, the phantom fields $\psi_1$, $\psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -0.8,\ Y(0) = -0.99,\ Z(0) = 0.805,\ U(0) = 0.99,\ \lambda_{\phi_1}(0) = 1/15,\ \lambda_{\phi_2}(0) = 1/15,\ \lambda_{\psi_1}(0) = -1/20,\ \lambda_{\psi_2}(0) = -1/20$. The figure shows that the equation of state of the split-quaternion field cold cross the phantom divide with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor. In Figs. (11) and (12), we plot the evolution of their equation of state, respectively.

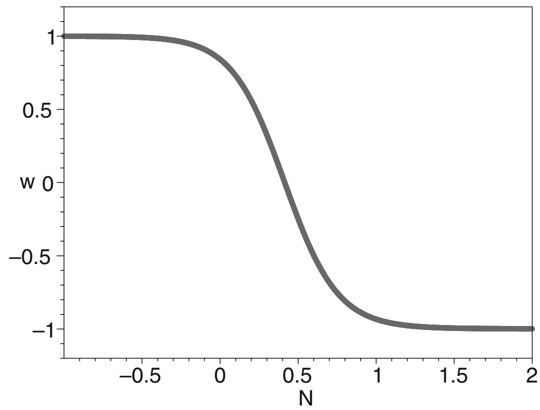


**Fig. 7** The evolution of the equation of state for the split-quaternion scalar when the kinetic energy of quintessence dominates over that of phantom (with the same strength of fields). It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts

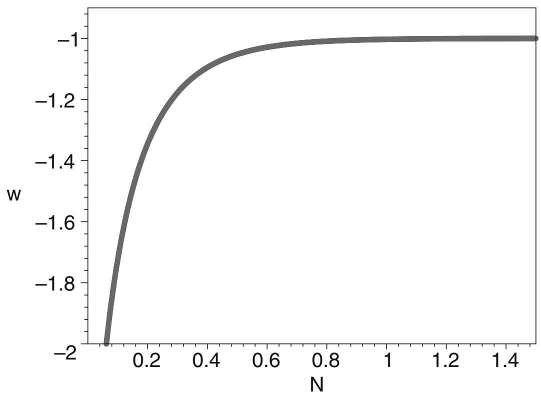


**Fig. 8** The evolution of the equation of state for the split-quaternion scalar when the kinetic energy of phantom dominates over that of quintessence (with the same strength of fields). The equation of state is always smaller than unit one

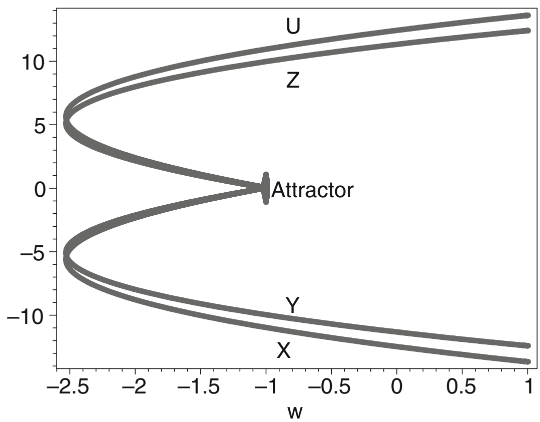


**Fig. 9** Evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1, \phi_2$, the phantom fields $\psi_1, \psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -1,\ Y(0) = -1.1,\ Z(0) = 1,\ U(0) = 1.1,\ \lambda_{\phi_1}(0) = 1/20,\ \lambda_{\phi_2}(0) = 1/20,\ \lambda_{\psi_1}(0) = -1/27,\ \lambda_{\psi_2}(0) = -1/20$. The figure shows that the equation of state of the split-quaternion field could cross the phantom divide with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor

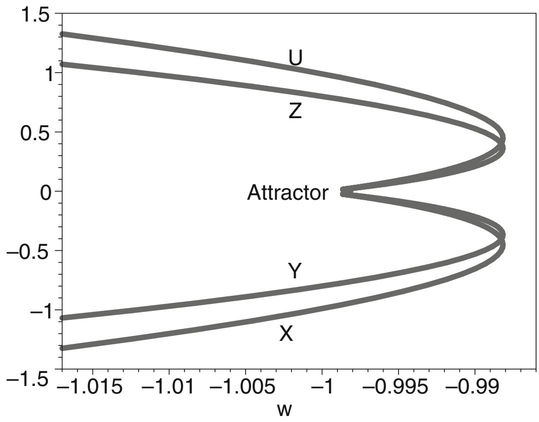


**Fig. 10** Evolution of the scaled velocities $X, Y, Z, U$ for the quintessence fields $\phi_1, \phi_2$, the phantom fields $\psi_1, \psi_2$, respectively, with respect to the equation of state $w$. The initial values are put by $X(0) = -0.8, Y(0) = -0.99, Z(0) = 0.805, \ U(0) = 0.99, \lambda_{\phi_1}(0) = 1/15, \lambda_{\phi_2}(0) = 1/15, \lambda_{\psi_1}(0) = -1/20, \lambda_{\psi_2}(0) = -1/20$. The figure shows that the equation of state of the split-quaternion field cold cross the phantom divide with the point $(X, Y, Z, U, w) = (0, 0, 0, 0, -1)$ as the attractor

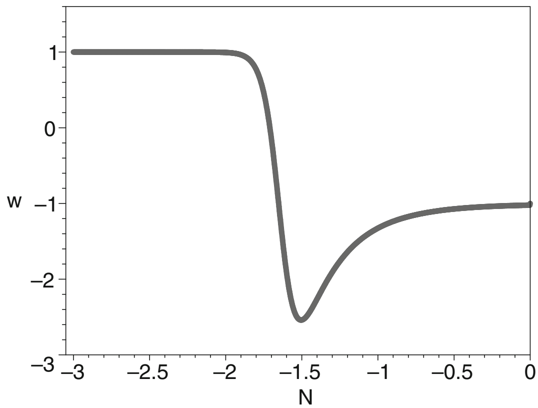


**Fig. 11** The evolution of the equation of state for the split-complex scalar corresponding Fig. (9). It behaves as the stiff matter at higher redshifts and a cosmological constant for the lower redshifts. The split-complex scalar can cross the phantom divide around the value of $N = -1.5$

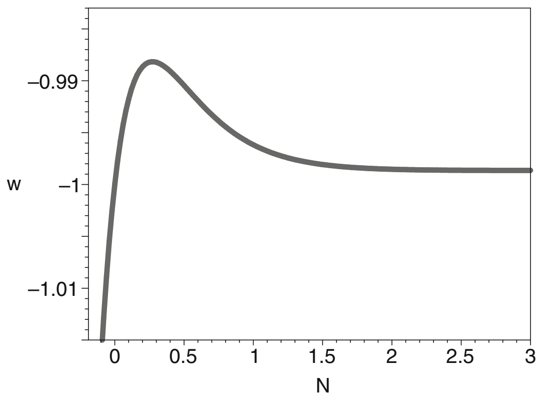


**Fig. 12** The evolution of the equation of state for the split-complex scalar corresponding Fig. (10). It behaves as the phantom matter at higher redshifts and a cosmological constant for the lower redshifts. The split-complex scalar can cross the phantom divide around the value of $N = 0.3$

In all, the detail of dynamics of the split-quaternion scalar field is closely related to the initial conditions on the fields.

### 4.3 Multi-field theory

In the proceeding sections, we have studied the cosmic evolution of the spilt-complex field and the split-quaternion field with the Lagrangian as follows, $\mathscr{L} = \frac{1}{2}\partial_\mu \Phi^I \partial^\mu \Phi_I + V$, Eqs. (17) and (45). In this subsection, we extend the expression of Lagrangian and show that they actually belong to the multi-field theories studied in the inflationary cosmology [57–61]. To this end, we define

$$X^{IJ} \equiv -\frac{1}{2}\partial_\mu \Phi^I \partial^\mu \Phi^J. \tag{53}$$

In the spirit of k-inflation [66], the very general Lagrangian is of the form

$$\mathscr{L} = K(X^{IJ},\ \Phi^I), \tag{54}$$

where $I = 1, \ldots, N$ labels the multiple fields. Here $\Phi_I$ can be split-complex field (N = 2), split-quaternion field (N = 4), split-octonion field (N = 8) and so on. The field indices $I$, $J$ are raised and lowered with the metric tensor $\eta_{IJ}$ and $\eta^{IJ}$ of the field space. If $\eta^{IJ}$ is replaced with the most general metric $\tilde{g}_{IJ}(\Phi^K)$, it is just the extensively studied multi-field theory in inflationary cosmology [57–61].

## 5 Linear perturbations

Up to now, we have explored the cosmic evolution of split-complex field and the split-quaternion field, respectively. In this section, we focus on the linear perturbations for the split-complex field. The extension of the result to split-quaternion and split-octonion field is straightforward.

We start from the action as follows (following the convention and notation in Ref. [57])

$$S = \int d^4x\sqrt{-g}\tilde{P}(Y, \Phi^I), \tag{55}$$

where

$$Y = \eta_{IJ}X^{IJ} + \frac{b\left(\Phi^K\right)}{2}\left(X^2 - X_I^J X_J^I\right), \tag{56}$$

where $\Phi^K$ is the split-complex field. $\eta_{IJ}$ is defined by Eq. (16). When $b = 0$, the Lagrangian, Eq. (17) is included as a particular case of $\tilde{P}(X,\ \Phi^K)$.

In order to study the evolution of linear perturbations in the background of FRW Universe, we expand the above action to second order, including both metric and scalar field perturbations. In the uniform curvature gauge, the perturbed split-complex field takes the form

$$\Phi^I(x,t) = \Phi_0^I(t) + Q^I(x,t), \tag{57}$$

where $Q^I$ denotes the field perturbations. In the following, we will usually drop the subscript "0″ on $\Phi_0^I$ and simply identify $\Phi^I$ as the fields in FRW Universe unless otherwise stated.

The second order action can be expressed as

$$\begin{aligned} S_{(2)} = \frac{1}{2}\int dtd^3xa^3\Big[&\left(\tilde{P}_{,Y}\eta_{IJ} + \tilde{P}_{,YY}\dot{\Phi}_I\dot{\Phi}_J\right)\dot{Q}^I\dot{Q}^J \\ &-\frac{1}{a^2}\tilde{P}_{,Y}\left[(1+bX)\eta_{IJ} - bX_{IJ}\right]\partial_i Q^I\partial^i Q^J \\ &-\bar{\mathcal{M}}_{IJ}Q^IQ^J + 2\tilde{P}_{,YJ}\dot{\Phi}_I Q^J\dot{Q}^I\Big], \end{aligned} \tag{58}$$

with the effective squared mass matrix

$$\begin{aligned} \bar{\mathcal{M}}_{IJ} = -\tilde{P}_{,IJ} &+ \frac{X\tilde{P}_{,Y}}{H}(\tilde{P}_{,YJ}\dot{\Phi}_I + \tilde{P}_{,YI}\dot{\Phi}_J) \\ &+\frac{X\tilde{P}_{,Y}^3}{2H^2}(1-\frac{1}{c_{ad}^2})\dot{\Phi}_I\dot{\Phi}_J \\ &-\frac{1}{a^3}\left[\frac{a^3}{2H}\tilde{P}_{,Y}^2\left(1+\frac{1}{c_{ad}^2}\right)\dot{\Phi}_I\dot{\Phi}_J\right]^{\cdot}. \end{aligned} \tag{59}$$

Here $\tilde{P}_{,Y}$ and $\tilde{P}_{,J}$ denote the derivative of $\tilde{P}$ with respect to $Y$ and $\Phi^J$. Dot denotes the derivative with respect to cosmic time $t$. $a$ and $H$ are the scale factor and Hubble parameter of the Universe, respectively. $c_{ad}$ is the sound speed for adiabatic perturbations defined by

$$c_{ad}^2 \equiv \frac{\tilde{P}_{,Y}}{\tilde{P}_{,Y} + 2X\tilde{P}_{,YY}}. \tag{60}$$

The sound speed squared of entropy perturbations is defined by

$$c_{en}^2 \equiv 1 + bX. \tag{61}$$

For Lagrangian, Eq. (17), we have

$$c_{ad} = c_{en} = 1, \tag{62}$$

and the second order action is simply

$$S_{(2)} = \frac{1}{2}\int dtd^3xa^3\left[\dot{Q}_I\dot{Q}^I - \frac{1}{a^2}\partial_i Q_I\partial^i Q^I - \bar{\mathcal{M}}_{IJ}Q^IQ^J\right], \tag{63}$$

where

$$\bar{\mathcal{M}}_{IJ} = \lambda^2 \Phi_I \Phi_J - \frac{1}{a^3}\left(\frac{a^3}{H}\dot{\Phi}_I \dot{\Phi}_J\right)^{\cdot}. \tag{64}$$

The equation of motion is given by

$$\ddot{Q}_I - \frac{1}{a^2}\partial_i \partial^i Q_I + \bar{\mathcal{M}}_{IJ} Q^J = 0. \tag{65}$$

In the gauge of $Q^2 = 0$, we have

$$\ddot{Q}^1 - \frac{1}{a^2}\partial_i \partial^i Q^1 + \lambda^2 \phi^2 Q^1 = 0. \tag{66}$$

The squared mass term is positive. Thus the perturbation to quintessence $Q^1$ is stable. On the other hand, in the gauge of $Q^1 = 0$, we have

$$\ddot{Q}^2 - \frac{1}{a^2}\partial_i \partial^i Q^2 - \lambda^2 \psi^2 Q^2 = 0. \tag{67}$$

In this case, the squared mass term is negative and the perturbation to phantom $Q^2$ is unstable.

In order to achieve a scale invariant power spectrum, we now decompose the perturbations into adiabatic and entropy part as follows, $Q^I = Q_\sigma e_\sigma^I + Q_s e_s^I$, where

$$e_\sigma^I = e_1^I = \frac{\dot{\Phi}^I}{\sqrt{P_{,X^{JK}} \dot{\Phi}^J \dot{\Phi}^K}}, \tag{68}$$

and $e_s^I$ is the unit vector orthogonal to $e_\sigma^I$. Use the conformal time $\tau = \int dt/a(t)$ and define the canonically normalized fields

$$v_\sigma \equiv \frac{a}{c_{ad}} Q_\sigma, \qquad v_s \equiv \frac{a}{c_{en}} Q_s, \tag{69}$$

one derive the equations of motion for $v_\sigma$ and $v_s$

$$\begin{aligned} v_s'' + \xi v_\sigma' + \left(c_{en}^2 k^2 - \frac{\alpha''}{\alpha} + a^2 \mu_s^2\right) v_s - \frac{z'}{z}\xi v_\sigma &= 0, \\ v_\sigma'' - \xi v_s' + \left(c_{ad}^2 k^2 - \frac{z''}{z}\right) v_\sigma - \frac{(z\xi)'}{z} v_s &= 0. \end{aligned} \tag{70}$$

Here the prime denotes the derivative with respect to $\tau$ and

$$\xi \equiv \frac{a}{\dot{\sigma}\tilde{P}_{,Y}c_{ad}}\left[(1+c_{ad}^2)\tilde{P}_{,s} - c_{ad}^2\dot{\sigma}^2\tilde{P}_{,Ys}\right],$$

$$\mu_s^2 \equiv -\frac{\tilde{P}_{,ss}}{\tilde{P}_{,Y}} - \frac{1}{2c_{ad}^2 X}\frac{\tilde{P}_{,s}^2}{\tilde{P}_{,Y}^2} + 2\frac{\tilde{P}_{,Ys}\tilde{P}_{,s}}{\tilde{P}_{,Y}^2},$$

$$z \equiv \frac{a\dot{\sigma}}{c_{ad}H}\sqrt{\tilde{P}_{,Y}}, \qquad \alpha \equiv a\sqrt{\tilde{P}_{,Y}}, \tag{71}$$

with

$$\dot{\sigma} \equiv \sqrt{2X}, \qquad \tilde{P}_{,s} \equiv \tilde{P}_{,I}e_s^I\sqrt{\tilde{P}_{,Y}}c_{en},$$

$$\tilde{P}_{,Ys} \equiv \tilde{P}_{,YI}e_s^I\sqrt{\tilde{P}_{,Y}}c_{en},$$

$$\tilde{P}_{,ss} \equiv \tilde{P}_{,IJ}e_s^I e_s^J\tilde{P}_{,Y}c_{en}^2. \tag{72}$$

In the limit of weak coupling, $\xi \simeq 0$, small effective mass, $\mu_s \simeq 0$ and slow-roll, $\dot{H} \simeq 0$, one have $z''/z = 2/\tau^2$ and $\alpha''/\alpha = 2/\tau^2$. So the equations of motion turn out to be

$$v_s'' + \left(c_{en}^2k^2 - \frac{2}{\tau^2}\right)v_s = 0,$$

$$v_\sigma'' + \left(c_{ad}^2k^2 - \frac{2}{\tau^2}\right)v_\sigma = 0. \tag{73}$$

Refs. [67] and [68] have ever shown that above equations lead to exact scale invariant power spectrum.

## 6 Conclusion and discussion

In conclusion, motivated by the mathematic theory of split-complex numbers, the split-quaternion numbers and split-octonion numbers, we have proposed the notion of split-complex scalar field, split-quaternion scalar field and the split-octonion scalar field. We note that our proposal of split scalar fields is not just a mathematical tool but rather it has physical information.

In the first place, the well-known quintessence and phantom fields could naturally emerge in these fields. So in order to construct a phantom field, one need not resort to quintessence by changing the sign of its kinetic term *by hand.* On the other hand, the conventional complex scalar field usually can not generate a phantom field by the decomposition, $\Phi = \phi + i\psi$ except for a non-canonical form of Lagrangian [54,55]. Secondly, if one require the split-complex scalar field obeys the symmetry of invariance under hyperbolic rotation, the split-complex field would restore to the Hessence field [54,55]. But if the symmetry of invariance is allowed to be broken, the split-complex scalar field would turn out to be the quintom field [11–17]. So compared to Hessence and quintom, they have rich physics. Thirdly, we find there is a conserved charge for the split-complex field. Therefore, the Coleman's Q-Balls-like solutions are expected to exist in the split-complex field which makes the field even more physical. Finally, by introducing the metric of field space, these split scalar fields fall into a subclass of the multi-field theories which are being studied in inflationary cosmology [57–61]. Also, following the usual procure of quantization of the complex scalar field [69], one could carry out the quantization of split-complex scalar field.

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