

Four-Loop Nonplanar Cusp Anomalous Dimension in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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The lightlike cusp anomalous dimension is a universal function that controls infrared divergences in quite general gauge theories. In the maximally supersymmetric Yang-Mills theory this function is fixed fully by integrability to the three-loop order. At four loops a nonplanar correction appears which we obtain for the first time from a numerical computation of the Sudakov form factor. Key ingredients are widely applicable methods to control the number-theoretic aspects of the appearing integrals. Our result shows explicitly that quadratic Casimir scaling breaks down at four loops.

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Introduction.—Quantum field theory is used throughout physics due to its unrivaled power to perform systematic computations. Connecting the calculated quantities to observed experimental variables, however, can be complicated. Perturbation theory for the theory of the strong nuclear force, quantum chromodynamics (QCD), for instance, can famously only be trusted on very small length scales [1,2]. The physical detectors, however, involve vastly larger length scales where the perturbation theory is guaranteed to break down. A central role in the analysis is played by universal functions, a prime example of which is known as the lightlike cusp anomalous dimension (CAD).

The universality of the CAD makes it appear in many contexts. For example, (i) by definition, refers to the anomalous dimension of a Wilson loop with a lightlike cusp [3,4]. (ii) It provides the leading infrared (IR) behavior of on-shell amplitudes, see, e.g., Ref. [5]. (iii) It determines the logarithmic growth of anomalous dimensions of high-spin Wilson operators that naturally appear in the operator product expansion description of deep inelastic scattering processes in QCD, see, e.g., Ref. [6]. (iv) It is related to the gluon Regge trajectory, and can be obtained from the so-called BFKL equation, see, e.g., Ref. [7]. (v) Via the AdS/CFT correspondence [8], cusped Wilson operators in the planar limit admit a dual description in terms of spinning strings in a curved background [9]. There is therefore ample direct physical motivation to compute this function.

In $\mathcal{N}=4$ super Yang-Mills (SYM) theory, the planar cusp anomalous dimension is known nonperturbatively via the so-called BES equation [10] based on ideas of the AdS/CFT correspondence and integrability [11]. Consistency with field-theory computations through to

four loops has been obtained at the planar level [12–14], and also confirmed by the strong coupling two-loop string computations [15]. There is an intriguing connection between $\mathcal{N}=4$ SYM and QCD results, via the maximal transcendentality property: the $\mathcal{N}=4$ anomalous dimensions correspond to the “leading-transcendentality” contribution in QCD [16,17]. Inspired by this, the three-loop $\mathcal{N}=4$ results were first obtained in Ref. [17] using the three-loop QCD results [18].

In gauge theories with matter fields in the adjoint representation the first correction beyond the planar limit enters into the CAD at four loops. This Letter contains to our knowledge the first computation of this quantity. In perturbation theory four-loop nonplanar form factor integrals have to date proven to be largely prohibitively complicated due to the presence of IR divergences. Nonplanar corrections are also hard to analyze directly within the AdS/CFT correspondence as they correspond to string loop corrections, and the role of integrability beyond the planar limit is unclear. Moreover, a quite general conjecture has been posed by extrapolating three-loop results that the nonplanar part of the CAD vanishes [19] in any theory. This is usually called quadratic Casimir scaling of the CAD, and plays an important role in the IR factorization in gauge theories [19–24]. This scaling is known to break down in planar $\mathcal{N}=4$ SYM theory at strong coupling [25], as well as through instanton effects [26]; see also Ref. [27] for a prediction of violation at four loops. In this Letter the approach will be strictly perturbative.

The minimal scatteringlike observable that contains the cusp anomalous dimension is the Sudakov form factor. In maximally supersymmetric Yang-Mills theory one can use

a correlator of a member of the stress-tensor multiplet with two on-shell massless states. The first computation of the two-loop correction to the Sudakov form factor in $\mathcal{N} = 4$ SYM theory appeared in Ref. [28]. The three-loop correction to the QCD result was studied in a series of papers [29–33]. In Ref. [34] these results were fine-tuned for the form factor in $\mathcal{N} = 4$ SYM theory to the three-loop order. The integrand for the four-loop Sudakov form factor in $\mathcal{N} = 4$ SYM theory was derived in Ref. [35] based on the duality between color and kinematics, and its reduction to master integrals was presented in Ref. [36]. Various other calculations of four-loop corrections in QCD were recently reported [37–42]. For the five-loop integrand in $\mathcal{N} = 4$ SYM theory see Ref. [43].

Review.—Form factor and cusp anomalous dimension: The Sudakov form factor involves only a single scale q^2 which is the Lorentzian norm of the sum of the two massless momenta, i.e., $q^2 = (p_1 + p_2)^2$ with $p_1^2 = p_2^2 = 0$. Dimensional analysis and maximal supersymmetry fix the form factor $\mathcal{F}^{(l)}$ at l loops to be given by

$$\mathcal{F}^{(l)} = \mathcal{F}^{\text{tree}} g^{2l} (-q^2)^{-l\epsilon} F^{(l)}, \quad (1)$$

where the coupling constant is normalized as $g^2 = [g_{\text{YM}}^2 C_A / (4\pi)^2] (4\pi e^{-\gamma_E})^\epsilon$. For a classical Lie group with Lie algebra $[T^a, T^b] = if^{abc} T^c$ and structure constants f^{abc} , gauge invariance dictates the color structure to be given by Casimir invariants. Up to three-loop order, only powers $(C_A)^l$ of the quadratic Casimir appear, for which $f^{acd} f^{bcd} = C_A \delta^{ab}$ holds. At four loops, the quartic invariant $d_{44} = d_A^{abcd} d_A^{abcd} / N_A$ appears in addition to $(C_A)^4$, with N_A the number of generators of the group and

$$d_A^{abcd} = \frac{1}{6} [f^{aa}{}_{\beta} f^{\beta b}{}_{\gamma} f^{\gamma c}{}_{\delta} f^{\delta d}{}_{\alpha} + \text{perms.}(b, c, d)]. \quad (2)$$

Starting from six loops, even higher group invariants appear, see, e.g., Ref. [35]. In $SU(N_c)$, $N_A = N_c^2 - 1$, $C_A = N_c$ and $d_{44} = N_c^2 / 24(N_c^2 + 36)$ hold. Without loss of generality, we will focus on $SU(N_c)$ group below.

The form factor has no ultraviolet (UV) divergences since the operator is protected, leaving only IR divergences. If dimensional regularization with $D = 4 - 2\epsilon$ is used to regulate the latter, $F^{(l)}$ is a purely numerical function of gauge group invariants and ϵ . This function is related to the cusp anomalous dimension $\gamma_{\text{cusp}}^{(l)}$ at l loops by [5,44–47],

$$(\log F)^{(l)} = - \left(\frac{\gamma_{\text{cusp}}^{(l)}}{(2l\epsilon)^2} + \frac{\mathcal{G}_{\text{coll}}^{(l)}}{2l\epsilon} + \text{Fin}^{(l)} \right) + \mathcal{O}(\epsilon). \quad (3)$$

At l loops the planar part $\propto N_c^l$ of $F^{(l)}$ has leading divergence $\propto 1/\epsilon^{2l}$. This function needs to be expanded down to ϵ^{-2} to extract the l -loop CAD, and also higher terms in the Laurent expansion in ϵ from lower-loop contributions are required. As mentioned, the first

occurrence of nonplanar (i.e., subleading-color) corrections to the CAD is at four loops, due to the appearance of the quartic Casimir invariant d_{44} . This invariant therefore breaks quadratic Casimir scaling explicitly. The relation between form factor and cusp anomalous dimension for the nonplanar part at four loops is

$$[F^{(4)}]_{\text{NP}} = - \frac{\gamma_{\text{cusp,NP}}^{(4)}}{(8\epsilon)^2} + \mathcal{O}(\epsilon^{-1}); \quad (4)$$

i.e., $[F^{(4)}]_{\text{NP}}$ has only a double pole in ϵ . Individual integrals that contribute to $[F^{(4)}]_{\text{NP}}$ will, however, typically show the full $1/\epsilon^8$ divergence. The general CAD is believed to be expressible as a rational-coefficient polynomial of Riemann Zeta values ζ_n , and their multi-index generalizations, such as multiple zeta values (MZVs) and Euler sums, see, e.g., Ref. [48]. MZVs are denoted by $\zeta_{n_1, n_2, \dots}$ and have a transcendental degree which is the sum of their indices, $\sum_i n_i$. At l loops, the planar CAD in $\mathcal{N} = 4$ SYM theory has uniform transcendentality degree $2l - 2$. At four loops, for instance, the planar CAD in $\mathcal{N} = 4$ SYM theory has been computed [12–14] to be

$$\gamma_{\text{cusp,P}}^{(4)} = -1752\zeta_6 - 64\zeta_3^2. \quad (5)$$

We will provide strong evidence that also the nonplanar CAD is of uniform transcendentality six at four loops.

In QCD, the known CAD has the same maximal transcendentality degree as in $\mathcal{N} = 4$, but also contains lower transcendentality degree constants. The maximal transcendentality coefficients match between planar $\mathcal{N} = 4$ and QCD, an observation known as the maximal transcendentality principle [16,17].

Integrands, integrals, integral relations.—The nonplanar part of the Sudakov form factor in $\mathcal{N} = 4$ SYM theory was obtained as a linear combination of a number of four-loop integrals in Ref. [35] using color-kinematics duality [49,50]. The integrals take the generic form

$$I = (q^2)^2 \int d^D l_1 \dots d^D l_4 \frac{N(l_i, p_j)}{\prod_{k=1}^{12} D_k}, \quad (6)$$

where D_i are propagators and the numerators $N(l_i, p_j)$ are quadratic polynomials of Lorentz products of the four independent loop and two independent external on-shell momenta. The explicit expressions of these integrals can be found in [35]. There are 14 distinct integral topologies that contribute to the nonplanar CAD, labeled (21)–(34) in Ref. [35], each with 12 internal lines. We will see below that only 10 of them, (21)–(30) as shown in Fig. 1, contribute to the nonplanar form factor if a basis of uniformly transcendental integrals is used.

Integrands are only identified up to terms that integrate to zero. Infinitesimal linear reparametrizations of the loop momenta generate such terms, which are known as integration-by-parts (IBP) identities [51,52]. With these

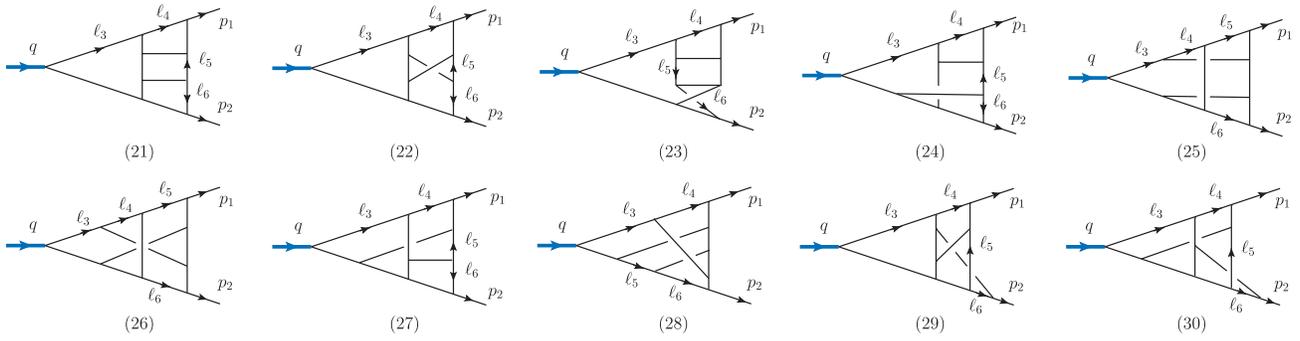


FIG. 1. Integral topologies that contribute to the nonplanar form factor at four loops in a basis of UT integrals.

identities the form factor was simplified in Ref. [36] using the Reduze code [53]. A particular subset of these relations, dubbed “rational IBP” relations and obtained in Ref. [54], will play an important role for the problem at hand. Note that integral relations due to graph symmetries are a particular subset of the rational IBP relations. Although simpler integrals emerged in Ref. [36] compared to Ref. [35], these have largely evaded integration so far due to their overwhelming complexity. The obstacle to computing the CAD is therefore to find a complete set of integrals which are simple enough to integrate.

Uniformly transcendental integrals.—The key idea is to find a representation of the four-loop form factor such that all integrals have uniform transcendentality if the dimensional regularization parameter ϵ is assigned transcendentality -1 . Such integrals will be referred to as UT integrals. Such a representation of the form factor makes manifest the expected maximal transcendentality property of $\mathcal{N} = 4$ SYM theory, and has been achieved at three loops in Ref. [34]. Crucially for our purposes, the UT integrals turn out to be much simpler to integrate than generic integrals.

Beyond explicit computation, there are various ways to prove a general integral is a UT integral: (1) A UT integral can be expressed in the so-called dLog form [55,56]. (2) A set of integral basis that are all UT integrals can lead to certain simple differential equations [57]. (3) The leading singularities, or, equivalently, the residues at all poles of a UT integral should be a constant [37,56,58], which is expected to be a sufficient and necessary condition.

We have obtained a dLog form, for instance, in topologies (21) and (23) for the integrals with numerator

$$[(l_3 - p_1)^2]^2 \quad (7)$$

each (see Fig. 1 for the labeling of momenta) and in topology (28) with numerator

$$(l_3 - l_4 - p_2)^2 (l_3 - p_1)^2. \quad (8)$$

The dLog form of the topology-(21) integral as well as its analytic result were also obtained in Ref. [37].

Finding a dLog form is, in general, a difficult problem. Moreover, the differential equations method is not directly

applicable for single-scale problems such as the Sudakov form factor (see Refs. [37,59] for a work around). The above third UT criterion, however, allows an algorithm. For this the four-dimensional loop momenta are parametrized as

$$l_1 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 q_1 + \alpha_4 q_2, \quad (9)$$

where p_1 and p_2 are the external on-shell momenta, and q_1, q_2 can be chosen as the two complex solutions to

$$q_i^2 = q_i \cdot p_j = 0 \quad \forall i, j \quad \text{and} \quad q_1 \cdot q_2 = -p_1 \cdot p_2. \quad (10)$$

With this, the integration volume changes to

$$\int d^4 l_1(\dots) = (p_1 \cdot p_2)^2 \int \prod_{i=1}^4 d\alpha_i(\dots). \quad (11)$$

In this so-called parametric form, the leading singularity is studied by computing subsequent residues for all occurring scalar parameters (16 at four loops). For this, one needs to choose an order of taking residues in the scalar parameters. If, after taking a residue in a particular parameter, one encounters other than a simple pole in a remaining parameter, the integral is not UT. In principle, there are $16! \sim 2 \times 10^{13}$ different orders in which the residues can be taken. We will explain below how we deal with this in practice.

Besides checking UT properties of single integrals, this procedure can be used to constrain the space of potential UT integrals of a given topology. Start with an ansatz of a linear combination of momentum products of mass dimension four. Given a set of integrals for a given topology one can see if one or more members fail the UT test for a particular randomly chosen order of residues. If a quadratic or higher pole is found, one can derive a linear constraint on the set of coefficients in the ansatz to evade the pole. Solving these constraints gives a smaller set of integrals. In the case at hand about a thousand such random checks typically yields a list of candidate UT integrals that do not easily yield further constraints. Through this strategy a set

of candidate integrals was obtained for each topology. We note that a related strategy was used in Refs. [37,56,58].

For ease of integration it is advantageous to have a form of the numerator as a product of two factors, both quadratic in momenta. To find linear combinations of UT candidate integrals with this property one writes a product-type ansatz and derives a set of quadratic-constraint equations. In some cases a solution may not exist or may not be useful. For the nonplanar four-loop form factor, this happens in topologies (26) and (25), respectively. Here the ansatz is widened by adding integrals with two fewer propagators, for instance, in topology (25) with numerator

$$[(p_1 - l_5)^2 + 2(l_4 - l_5)^2 + (l_3 - l_4)^2 - (l_3 - l_5)^2 - (p_1 - l_4)^2]^2 - 4(l_4 - l_5)^2(p_1 - l_3 + l_4 - l_5)^2. \quad (12)$$

Beyond integrals for which a dLog form was obtained, all UT candidates have passed a significant number of residue checks, e.g., the two most complicated ones are in topology (26) and have passed 10^5 such checks. Moreover, as discussed in the next section, the precision of the explicit numerical computation permits a conversion of all results up to $\mathcal{O}(\epsilon^{-4})$ to analytic expressions which exhibit uniform transcendentality. Taken together, these features provide strong evidence that all candidates are indeed UT integrals.

Having obtained a set of UT candidate integrals, one needs to express the form factor as a linear combination of them. Since the form factor is expected to be uniformly transcendental itself, this linear combination must involve only rational numbers—not ϵ . The aforementioned “rational IBP” relations obtained in Ref. [54] prove extremely useful for this task.

The first result of this Letter is that the nonplanar part of the four-loop form factor obtained in Ref. [36] can be expressed in terms of 23 UT candidate integrals, providing strong evidence for maximal transcendentality in the nonplanar sector. The full list of these integrals and the appropriate linear combination is given in the Supplemental Material [60], see also Ref. [61]. Interestingly, for topologies (31)–(34), no UT candidate integrals could be found.

Numerical integration.—A variety of techniques exist to compute loop integrals both analytically and numerically, see, e.g., Ref. [62] for an overview. Here, mainly sector decomposition and Mellin-Barnes (MB) integration have been used, as well as an exact integral result of topology (21) [37].

Two computer implementations of sector decomposition [63] have been used: mostly FIESTA [64–67], with cross-checks for simpler integrals using SECDEC [68–70]. An important empirical observation within the sector decomposition approach is the feature that UT integrals usually generate considerably fewer integration terms compared to non-UT siblings of comparable complexity, even though, for instance, single-sector contributions are not UT separately. The numerical integration with FIESTA is performed using

the VEGAS algorithm [71] as implemented in the CUBA library [72]. When using SECDEC, the CUHRE and DIVONNE algorithms are applied.

A second integration technique applied here is MB integration. Several automated tools exist, such as MB tools [73,74] and AMBRE [75–77]. Given an integral of the form (6), the main challenge is to derive valid MB representations for crossed four-loop topologies. Having achieved the latter, the obtained representations are in many cases too high-dimensional to be integrated. Still, efficient MB representations for a subset of (planar and crossed) integrals could be derived by means of a hybrid of the loop-by-loop approach and using the \mathcal{F} and \mathcal{U} graph polynomials (see Ref. [61] for details). If an efficient MB representation is available, the numerical precision of the integral is typically 3 to 4 orders of magnitude better compared to sector decomposition.

Before presenting our results, a thorough investigation of the numerical uncertainties is in order. In sector decomposition, several runs with different integration settings were performed. We observe that error bars reported by FIESTA scale with $1/\sqrt{N_s}$, with N_s the number of sampling points (typically in the range of a few up to several 100 millions). Moreover, we observe that fluctuations upon increasing N_s are well within the reported error bars. For the leading nontrivial poles, $\epsilon^{-\{8,6,5,4\}}$, the numerical precision in each UT integral is high enough to write the number as a *small* rational multiple of $\{1, \zeta_2, \zeta_3, \zeta_4\}$, respectively. The reliability of the error bars reported by FIESTA was checked by comparing to MB and (semi-)analytic results where available. Altogether, this comparison involves more than seventy data points. The fluctuations for all integrals used are within the reported FIESTA uncertainties, in many cases by over an order of magnitude.

The uncertainties will be conservatively interpreted as the standard deviation of a Gaussian distribution, and errors are added in quadrature. Our results show no evidence of systematic effects in the FIESTA errors: deviations to other results include both positive and negative signs. A hypothetical systematic error will be modeled by adding uncertainties from individual UT integrals linearly. In conclusion, there is no need to manually inflate the reported uncertainties. Further details will be provided in Ref. [61].

Results.—Our results for the nonplanar four-loop form factor up to an overall factor are summarized in Table I. The first 6 orders in the ϵ expansion must vanish by Eq. (4). The ϵ^{-7} coefficient should vanish for each integral, and indeed does well within error bars. The coefficients of $\epsilon^{-\{8,6,5,4,3\}}$ can be nonzero in individual integrals but must cancel in the linear combination of the form factor, which is indeed the case, both using direct numerics (see Table I) as well as the obtained analytic expressions.

The nontrivial coefficient for the nonplanar form factor at order ϵ^{-2} yields the sought-after nonplanar four-loop CAD:

TABLE I. Nonplanar form-factor result and errors. The ϵ^{-8} entry is afflicted with a rounding error.

ϵ order	-8	-7	-6	-5	-4	-3	-2
result	-3.8×10^{-8}	$+4.4 \times 10^{-9}$	-1.2×10^{-6}	-1.2×10^{-5}	$+3.5 \times 10^{-6}$	+0.0007	+1.56
uncertainty	$\pm 7.9 \times 10^{-9}$	$\pm 5.7 \times 10^{-7}$	$\pm 1.0 \times 10^{-5}$	$\pm 1.2 \times 10^{-4}$	$\pm 1.5 \times 10^{-3}$	± 0.0186	± 0.21

$$\gamma_{\text{cusp, NP}}^{(4)} = -3072 \times (1.56 \pm 0.21) \frac{1}{N_c^2}, \quad (13)$$

where $3072 = 2 \times 24 \times 64$ is the normalization factor arising from the permutational sum, the color factor [35], and the denominator of Eq. (4), respectively. The significance of a deviation from zero is 7.4σ . This is the second major result of this Letter. Adding individual uncertainties linearly would yield 1.56 ± 0.62 , still significantly nonzero.

Compared to the planar result (5): $\gamma_{\text{cusp, P}}^{(4)} \sim -1875$, we see that the nonplanar CAD has the same sign. If $N_c = 3$ is used, its central value becomes $\gamma_{\text{cusp, NP}}^{(4)} \sim -532$: i.e., the planar contribution is a factor of 3–4 larger.

Discussion.—In this Letter we present the first computation of the nonplanar correction to the cusp anomalous dimension in the maximally supersymmetric Yang-Mills theory, which starts at four loops.

While integrating a generic set of four-loop form factor master integral remains quite challenging, a key idea leading to the present result is to express the form factor in a basis of uniformly transcendental integrals, which can be constructed algorithmically. While a generic integral contains mixed transcendentality degrees, the UT integrals contain only numbers of fixed (maximal for $\mathcal{N} = 4$ SYM theory) transcendentality in each order of the ϵ expansion. What is interesting and deserves further study is the empirical observation that such simplicity is inherited in sector decomposition, where each sector is no longer UT separately. Once a good set of (candidate) UT basis integrals is determined, the full form factor can be expressed in this basis by using a simple subset of the IBP relations or, in principle, directly by unitarity cuts. In the case at hand, our results provide strong evidence for the maximal transcendentality of the four-loop non-planar form factor in $\mathcal{N} = 4$ SYM theory. In particular, this implies that the four-loop CAD can be written as a rational linear sum of weight-six transcendental numbers.

The UT basis finding algorithm is widely applicable. An immediate interesting application of already obtained results would be to four-loop propagator integrals in QCD, see, e.g., Ref. [78]. A UT basis in QCD would always involve prefactors with non-negative powers of ϵ , simplifying computations potentially drastically; see Ref. [79] for similar ideas.

Based on the UT basis, a numerical computation has also yielded the first information on the value of the nonplanar

CAD in $\mathcal{N} = 4$, which is statistically significantly nonzero. In particular, our result shows that quadratic Casimir scaling breaks down at four loops in this theory and is therefore not expected to hold in any other theory such as QCD. Direct further research directions include improving precision at order $1/\epsilon^2$ and computing further orders in the ϵ expansion, which contain the so-called collinear anomalous dimension. Computing the four-loop nonplanar CAD analytically, also in other theories such as QCD, is a prime further goal.

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Note added.—During the review of this Letter Refs. [80,81] have appeared, which report violation of quadratic Casimir scaling at four loops in QCD.

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