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Commun. Theor. Phys. **68** (2017) 323–328 Vol. 68, No. 3, September 1, 2017

Tunnelling Phenomenon near an Apparent Horizon in Two-Dimensional Dilaton Gravity*

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(Received April 27, 2017; revised manuscript received May 23, 2017)

Abstract *Based on the definition of the apparent horizon in a general two-dimensional dilaton gravity theory, we analyze the tunnelling phenomenon near the apparent horizon. In this theory the definition of the horizon is very different from those in higher-dimensional gravity theories. By using the Hamilton–Jacobi method, the spectrum of the radiation is obtained and the temperature of the radiation is read out from this spectrum. The temperature is proportional to the surface gravity of the apparent horizon as usual. Besides, in stationary cases we calculate the spectrum by using Parikh and Wilczek's null geodesic method and the result conforms to that obtained by using the Hamilton–Jacobi method. This is expected since the Hamilton–Jacobi method applies to generic spacetimes, including stationary ones.*

PACS numbers: 04.02.-q, 04.50.Kd **DOI:** 10.1088/0253-6102/68/3/323

Key words: two-dimensional dilaton gravity, apparent horizon, tunnelling phenomenon

1 Introduction

From analyzing the quantum field theory on a fixed curved background spacetime, Hawking showed that a black hole behaves like a black body, radiating with a temperature proportional to the surface gravity of the black hole and an entropy proportional to the area of the cross section of the event horizon.[1−2] Some nice reviews and surveys on this can be found in Refs. [3–7]. In Hawking's work, he suggested a heuristic picture to explain the process of the radiation, i.e., a pair of virtual particles are created inside the horizon and the one with positive energy tunnels though the event horizon and materializes outside the horizon while the other one with negative energy is absorbed by the black hole making the mass of the black hole decrease. Then the escaped particles run to the infinity, visible to distant observers appearing as Hawking radiation.[8] Because of the radiation, the black hole loses energy and therefore shrinks, evaporating away to an unknown fate. But the actual derivations of Hawking radiation did not correspond directly to the heuristic picture.[1−2,8−9] And in the original derivations, a stationary black hole with an event horizon is essential.

The idea is really visual and intuitively appealing. Since the Hawking radiation relates the theory of general relativity to quantum field theory and statistical thermodynamics, it is generally believed that a deeper understanding of Hawking radiation may shed some lights on seeking the underlying quantum gravity.[10] But unfortunately, there are two missing points in Hawking's actual derivation of the radiation. The first is the stationary black hole. With particle emission, the black hole cannot be stationary. Moreover, stationary black holes are very rare in the universe. Actual black holes are always dynamical for the accretion of matter or energy and the back-reaction of the radiation if it really exists. The second is the requirement of an event horizon which depends on the global structure of the spacetime and there are some practical issues, which can not be solved easily.[11] So the event horizon could either not exist or be a meaningless concept.[12−13] In the past decades, some semiclassical methods have been proposed in order to treat Hawking radiation as a tunnelling effect near the horizon by Refs. [14–19]. Two principal implementations of the tunnelling approach are the null geodesic method[20] and the Hamilton–Jacobi method.[21] They apply to a large class of dynamical spacetimes with only local horizons.[22−28] The key to the tunnelling method is the computation of the imaginary part of the single-particle action by integrating along a null path crossing the horizon. Applying the WKB approximation, one finds that the tunnelling probability is proportional to $\exp(-2\mathrm{Im}S)$, where S is the classical action to the leading order in $\hbar$ (here $\hbar = 1$). Expanding the imaginary part of the action in terms of the particle energy, at the linear order, we get the Hawking radiation spectrum and with the quadratic term (the quadratic term

*Supported by the National Natural Science Foundation of China under Grant Nos. 11622543 and 11235010, and by the Open Project Program of Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, China (No. Y5KF161CJ1)

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appears in the computation of the null geodesic method while in that of the Hamilton–Jacobi method it seems absent), the back-reaction effect can be recovered.[20,29]

Both the null geodesic and Hamilton–Jacobi tunnelling methods can be applied to a wide class of spacetimes and are in agreement with the visual picture suggested by Hawking. They are shown to be equivalent at least in the stationary case.[29−31] The Hamilton–Jacobi method is more powerful since it is quite convenient for treating truly dynamical black holes.[29,32] But there is still an important missing point when one deals with the non-spherically symmetric dynamical case, i.e., the absence of an extension of Kodama–Hayward theory.[27,29] In axis-symmetric dynamical spacetimes, a method called generalised tortoise coordinate transformation (GTCT) is developed by authors in Refs. [33–34] and the Hawking radiation is related to the event horizon. In spherically symmetric cases, the Kodama vector[35] instead of the original Killing vector can be used to define the direction of the Hamilton flow and the horizons would be some quasi-local horizons whose definitions closely rely on the concept of Hayward's trapping horizon.[36−37] In recent years, Senovilla and Torres have provided a general formula which could be used to analyze the phenomenon of tunnelling in arbitrary spacetimes with marginally trapped surfaces (MTSs) by using the dual expansion vector to replace the Kodama vector.[38] In this paper, we will use this method to analyze the tunnelling phenomenon in a general two-dimensional dilaton gravity.

Some important technical complications in dealing with the basic questions of quantum gravity in higher dimensions make the treatment of these basic questions extremely difficult. Therefore, a rich literature has been developed on lower-dimensional models of gravity in the past years. For instance, the two-dimensional dilaton gravity has been widely studied over the past twenty years. This kind of gravity can be obtained from the spherically symmetric reduction of Einstein gravity theory in higher dimensions or by eliminating Weyl anomaly on string world sheet, e.g. CGHS model[39] (for a nice review, see Resf. [40–41]). A lot of black hole solutions (e.g. Ref. [42]) and cosmological solutions have been found in these theories and most of them become general dynamical solutions when some matter fields are introduced which will make the situations become complicated. In the paper,[43] the definition of an apparent horizon is provided in the general two-dimensional dilaton gravity theory and then the mechanics of the horizon is constructed by introducing a quasi-local energy and a Kodama-like vector field. As we can see, their definition of the horizon is very general and quasi-local. Therefore, it is very interesting and important to make sure whether the radiation also exists and to study its properties if so.

This paper is organized as follows: In Sec. 2, we briefly review the definition of the apparent horizon in the general two-dimensional dilaton gravity and give some simple discussions of its properties. In Sec. 3, we use the Hamilton–Jacobi method to analyze the tunnelling phenomenon in the general cases in the two-dimensional dilaton gravity. We then use the null geodesic method to deal with a specific case and compare the results of the two situations in Sec. 4. Finally, the conclusions and discussions are in Sec. 5. We will use the unit where $c = G = k_B = \hbar = 1$ in this paper. Latin indices as $a\,,b$ are used to denote the abstract indices and Greek indices as $\mu\,,\nu$ are used to denote the components of a tensor.

2 Apparent Horizon in Two-Dimensional Dilaton Gravity

In this section we will follow the discussions in Ref. [43]. The action of a general two-dimensional dilaton gravity can be expressed as[41]

$$I = \int \mathrm{d}^2x\sqrt{-h}[\Phi R + U(\Phi)D^a\Phi D_a\Phi + V(\Phi) + \mathscr{L}_m]\,, \quad (1)$$

where h_{ab} is the spacetime metric, h its determinant, R the Ricci scalar, Φ the so-called dilaton field, $U(\Phi)$ and $V(\Phi)$ arbitrary functions of Φ. $V(\Phi)$ is a potential and $\mathscr{L}_m$ is the matter Lagrangian. The equation of motion for the dilaton field Φ is

$$R - U'(\Phi)D_a\Phi D^a\Phi + V'(\Phi) - 2U(\Phi)\Box\Phi + \mathscr{T}_m = 0\,, \quad (2)$$

where D_a is the derivative operator compatible with h_{ab}, and $\Box = D_aD^a$. The prime stands for the derivative with respect to Φ: $\mathrm{d}/\mathrm{d}\Phi$, while the scalar $\mathscr{T}_m$ is defined as

$$\mathscr{T}_m := \frac{\partial\mathscr{L}_m}{\partial\Phi} - D_a\frac{\partial\mathscr{L}_m}{\partial(D_a\Phi)} + \cdots \quad (3)$$

The equation of motion for the metric h_{ab} is

$$U(\Phi)D_a\Phi D_b\Phi - \frac{1}{2}U(\Phi)D^c\Phi D_c\Phi h_{ab} - D_aD_b\Phi + \Box\Phi h_{ab} - \frac{1}{2}V(\Phi)h_{ab} = T_{ab}\,, \quad (4)$$

where T_{ab} is the energy momentum tensor of the matter field.

Assume $\{\ell^a, n^a\}$ is a null frame in the spacetime and the metric can be expressed as

$$h_{ab} = -\ell_a n_b - n_a\ell_b\,, \quad (5)$$

while ℓ^a and n^a are two future directed null vector fields which are globally defined on the spacetime and satisfy $\ell_a n^a = -1$. Also we assume ℓ_a and n^a are outer pointing and inner pointing respectively. Now we introduce an important vector field $\phi^a = D^a\Phi$. In the null frame, it is easy to see

$$\phi_a\phi^a = D_a\Phi D^a\Phi = -2\mathcal{L}_\ell\Phi\mathcal{L}_n\Phi\,, \quad (6)$$

where $\mathcal{L}_X$ is the Lie derivative along any vector denoted by X. Considering the causality of this vector field, the spacetime can be divided into several parts, and in each part the vector field ϕ^a is either spacelike or timelike and on the boundary of any part it is null, and this boundary can be defined as a kind of horizon. The definitions are[43]

future outer horizon : $\mathcal{L}_\ell\Phi = 0, \quad \mathcal{L}_n\Phi < 0, \quad \mathcal{L}_n\mathcal{L}_\ell\Phi < 0\,;$
future inner horizon : $\mathcal{L}_\ell\Phi = 0, \quad \mathcal{L}_n\Phi < 0, \quad \mathcal{L}_n\mathcal{L}_\ell\Phi > 0\,;$
past outer horizon : $\mathcal{L}_n\Phi = 0, \quad \mathcal{L}_\ell\Phi > 0, \quad \mathcal{L}_\ell\mathcal{L}_n\Phi > 0\,;$
past inner horizon : $\mathcal{L}_n\Phi = 0, \quad \mathcal{L}_\ell\Phi > 0, \quad \mathcal{L}_\ell\mathcal{L}_n\Phi < 0\,.$

We will focus on the future outer horizons in the following discussions. From the discussion in the introduction we can see that in higher dimensional spaetimes, the radiation or particle creation is related to some local region around, which some key vector field changes its causality from temporal to spatial. The key vector field is the Killing vector field in stationary spacetimes or the Kodama vector field in spherically symmetric dynamical spacetimes. In more general spacetimes which are not spherically symmetric it is the dual expansion vector field:

$$H^a \equiv -\theta_n \ell^a + \theta_\ell n^a\,, \tag{7}$$

where θ_ℓ and θ_n are the expansion scalars of the outgoing and ingoing null congruences normal to a codimension-2 spacelike surface respectively.[38] At first sight, the expansion scalars and the vector (7) can not be defined in the general two-dimensional dilaton gravity theory since the codimension-2 surface shrinks to a point in two dimensions. However, thanks to the dilaton field Φ, if we regard $\mathcal{L}_\ell\Phi$ and $\mathcal{L}_n\Phi$ as the counterparts in two dimensions of θ_ℓ and θ_n respectively, we can define the dual expansion vector field in this case as

$$H^a \equiv -(\mathcal{L}_n\Phi)\ell^a + (\mathcal{L}_\ell\Phi)n^a\,, \tag{8}$$

which is dual to the vector field ϕ^a. The dual expansion vector field H^a is also parallel to the Kodama vector field K^a which is used to define the surface gravity in Ref. [43]. The region with $\mathcal{L}_\ell\Phi < 0$ and $\mathcal{L}_n\Phi < 0$ can be called the trapped region of the spacetime, and the boundary of the region therefore can be defined as a horizon. From this point of view, the definition of the horizon is natural and reasonable. Also, the future outer horizon has the traditional meaning, i.e. the boundary of a region from which the light cannot escape along a classical trajectory. With the expression of the dual expansion vector (8), we can see that

$$h_{ab}H^aH^b = 2\theta_\ell\theta_n = 2\mathcal{L}_\ell\Phi\mathcal{L}_n\Phi\,. \tag{9}$$

So the dual expansion vector is timelike outside the trapped region, spacelike inside the trapped region and null on the horizon, which is the desirable property of the dual expansion vector field.

3 General Tunnelling Process

In this section we will use the Hamilton–Jacobi tunnelling method to analyze the tunnelling phenomenon associated to the future outer horizon in the two-dimensional dilaton gravity theory. We will see that the tunnelling phenomenon will exist near the apparent horizon (future out horizon) as long as the spacetime has an apparent horizon defined above but without any further assumption.

As the authors in Ref. [38], we can define a vector

$$\xi^a \equiv \mathrm{e}^Q H^a\,, \tag{10}$$

where $Q' = -U$. This vector on the one hand can be used to define the apparent horizon and on the other hand will coincide with the Kodama-like vector in Ref. [43] when we choose the Eddington–Finkelsten gauge. For a given massless particle one can define the energy ω by

$$\omega = -\xi^a D_a S\,. \tag{11}$$

We also assume that the particle's action satisfies the relativistic Hamilton–Jacobi equation,

$$h^{ab} D_a S D_b S = 0\,. \tag{12}$$

Using the double null metric (5) and taking Eqs. (10), (11), (12) into consideration together, we get

$$0 = (\ell^a\partial_a S)(n^b\partial_b S)\,, \tag{13}$$

$$\omega = e^Q(\mathcal{L}_n\Phi)(\ell^a D_a S) - \mathrm{e}^Q(\mathcal{L}_\ell\Phi)(n^a D_a S)\,, \tag{14}$$

from which we can get two solutions $\ell^a D_a S = 0$ or $n^a D_a S = 0$.

As we can see the parameter of ℓ^a is not continuous at the horizon, so the outgoing particles can not cross the horizon along the trajectories of the outgoing null curves. When the particles cross the horizon along the trajectories of the ingoing curves, the derivative of the action along ℓ^a is unchanged because n^a and ℓ^a are two independent vectors in two-dimensionanl spacetime. So we get the solution which corresponds to outgoing mode

$$\ell^a D_a S = 0\,, \tag{15}$$

$$n^a D_a S = -\frac{\omega}{\mathrm{e}^Q(\mathcal{L}_\ell\Phi)}\,. \tag{16}$$

From the definition of the future out horizon we can know that $\mathcal{L}_\ell\Phi$ vanishes at the horizon. This informs us that $n^a D_a S$ diverges when the particles cross the horizon. We can say, the horizon acts as a classical forbidden barrier for the particles. The particles travel along classically forbidden trajectories by starting just behind the horizon onwards to infinity so that the particles must travel back in time.[29] Then the imaginary part of the action for an outgoing particle traveling from a point denoted by "in" inside the horizon to another point denoted by "out" outside the horizon is

$$\mathrm{Im}S = \mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} \mathrm{d}S = \mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} \delta^\nu_\mu \partial_\nu S\, \mathrm{d}x^\mu$$

$$= \mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} -[(n^\nu\partial_\nu S)\ell_\mu\,\mathrm{d}x^\mu + (\ell^\nu\partial_\nu S)n_\mu\,\mathrm{d}x^\mu]$$
$$= \mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} \frac{\mathrm{e}^{-Q}\omega}{\mathcal{L}_\ell\Phi}\ell_\mu\,\mathrm{d}x^\mu\,. \tag{17}$$

It is obvious that the action S has a pole at the horizon and the integration is divergent. We will regularise the divergence according to Feynman's iϵ-prescription. Introducing λ as the parameter of the tangent vector of the ingoing null curve i.e. $n^a = (\mathrm{d}/\mathrm{d}\lambda)^a$, then the integration 1-form can be expressed as

$$\ell_\mu\,\mathrm{d}x^\mu = -\mathrm{d}\lambda\,. \tag{18}$$

From the discussion above, only a small segment crossing the horizon will make sense. And recall that $\mathcal{L}_\ell\Phi = 0$ on the horizon, so in the neighbourhood of the horizon denoted by "h" it can be expressed as

$$\mathcal{L}_\ell\Phi \approx \frac{\mathrm{d}(\mathcal{L}_\ell\Phi)}{\mathrm{d}\lambda}\Big|_h(\lambda-\lambda_h)\,, \tag{19}$$

where λ_h is the parameter value of the point where the ingoing null curve crosses the horizon. Now, inserting Eqs. (18) and (19) into Eq. (17) and using the Feynman's iϵ-prescription, the imaginary part of the action is (for more details one can refer to Ref. [29])

$$\mathrm{Im}S = -\mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} \frac{\omega\,\mathrm{e}^{-Q}}{\mathrm{d}(\mathcal{L}_\ell\Phi)/\mathrm{d}\lambda|_h(\lambda-\lambda_h-\mathrm{i}\epsilon)}\mathrm{d}\lambda$$
$$= \frac{\pi\omega}{\kappa}\,, \tag{20}$$

where κ is defined as

$$\kappa = -\,\mathrm{e}^Q\frac{\mathrm{d}(\mathcal{L}_l\Phi)}{\mathrm{d}\lambda}\Big|_h = -\,\mathrm{e}^Q\mathcal{L}_n\mathcal{L}_\ell\Phi|_h\,. \tag{21}$$

From the discussion in Ref. [43], we have the equation

$$\mathcal{L}_n\mathcal{L}_\ell\Phi = -\kappa_{(n)}(\mathcal{L}_\ell\Phi) - (1/2)\Box\Phi\,, \tag{22}$$

where $\kappa_{(n)}$ is a scalar defined by $\kappa_{(n)} = -\ell_a n^b D_b n^a$. As we treat the future outer horizon here, we have

$$\Box\Phi > 0\,. \tag{23}$$

Then κ can be expressed as

$$\kappa = \frac{1}{2}\,\mathrm{e}^Q\Box\Phi\,, \tag{24}$$

which is nothing but the surface gravity in Ref. [43] and always positive on the future outer horizon. Meanwhile, the tunnelling probability becomes

$$\Gamma \propto \exp\Big(-\frac{2\pi\omega}{\kappa}\Big)\,. \tag{25}$$

Combining with the Boltzmann factor $\exp(-\omega/T)$ of the thermal radiation, we get

$$T = \frac{\kappa}{2\pi}\,, \tag{26}$$

which is exactly the famous relation between temperature and surface gravity in the Hawking radiation.

Therefore, we have proven that the tunnelling phenomenon still exists in the two-dimensional dilaton gravity theory as long as there is a local horizon in the spacetime. The temperature of the radiation and the surface gravity of the horizon satisfy the usual identity.

4 The Stationary Case

In this section, we will use the null geodesic method to study the tunnelling phenomenon in the two-dimensional stationary spacetime as the null geodesic method is very inconvenient to treat truly dynamical black holes.[29] We consider the vacuum case of Eq. (4). In Eddington–Finkelsten gauge, a general solution has a simple form[43]

$$\mathrm{d}s^2 = -\,\mathrm{e}^Q(w-2m)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r\,, \tag{27}$$

where functions $Q(r)$, $w(r)$ and r satisfy the relations[29]

$$U = -Q'\,,\quad V = \mathrm{e}^{-Q}w'\,,\quad \mathrm{d}r = \mathrm{e}^Q\mathrm{d}\Phi\,, \tag{28}$$

and m is the black hole mass. When one uses the null geodesic method, it is usually convenient to choose the Painlevé–Gullstrand gauge.[44] If we use a function f to denote the metric component h_{vv} then the Painlevé time t can be defined by

$$\mathrm{d}t = \mathrm{d}v - f^{-1}(1-\sqrt{1-f})\mathrm{d}r\,. \tag{29}$$

The solution (27) can be expressed as

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + 2\sqrt{1-f}\,\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2\,, \tag{30}$$

which is a stationary metric and the apparent horizon is defined by

$$f = \mathrm{e}^Q(w-2m) = 0\,. \tag{31}$$

This can be regarded as a special case of the general definition in Sec. 2. In this stationary spacetime, the vector (10) coincides with the Kodama-like vector field in Ref. [43] and becomes

$$\xi^a = \Big(\frac{\partial}{\partial t}\Big)^a\,, \tag{32}$$

which is nothing but the Killing vector field of the spacetime. The energy (11) now is invariant along the particle's propagating path, and it is just the particle energy measured by the observers at infinity when the spacetime is asymptotically flat. The outgoing null geodesic is given by

$$\dot{r} = 1 - \sqrt{1-f}\,, \tag{33}$$

where the overdot stands for the derivative with respect to t. The imaginary part of the action of an outgoing particle crossing the horizon can be expressed as

$$\mathrm{Im}S = \mathrm{Im}\int_{\mathrm{in}}^{\mathrm{out}} p_r\,\mathrm{d}r = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{p_r}\mathrm{d}p'_r\,\mathrm{d}r\,, \tag{34}$$

where the prime is a symbol of integration variable, which should not be confused with the derivative with respect to Φ. In order to calculate the integral, we change the integration variable from momentum to energy by using the Hamilton's equation $\dot{r} = \mathrm{d}H/\mathrm{d}p_r|_r$, where H is the generator of Painlevé time.[45] Then we get

$$\mathrm{Im}S = \mathrm{Im}\int_m^{m-\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\frac{\mathrm{d}r}{\dot{r}}\,\mathrm{d}H$$

$$= \mathrm{Im}\int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{\mathrm{d}r}{1-\sqrt{1-\mathrm{e}^{Q}(w-2m+2\omega')}}(-\mathrm{d}\omega')$$
$$= \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} \frac{(-\omega)\mathrm{d}r}{1-\sqrt{1-\mathrm{e}^{Q}(w-2m)}} + O(\omega^2)$$
$$= \mathrm{Im}\int_{\Phi_{\rm out}}^{\Phi_{\rm in}} \frac{\omega\,\mathrm{e}^{Q}\mathrm{d}\Phi}{1-\sqrt{1-\mathrm{e}^{Q}(w-2m)}} + O(\omega^2)\,. \quad (35)$$

The third step comes from the fact that $\omega \ll m$ and we use the definition $\mathrm{d}r = \mathrm{e}^{Q}\mathrm{d}\Phi$ in the last step. To the first order in ω, the integration also has a pole and we can regularise its divergence with the Feynman's iϵ-prescription as before. The final result is

$$\mathrm{Im}S = \frac{2\pi\omega}{w'} + O(\omega^2)\,, \quad (36)$$

where the prime stands for the derivative with respect to Φ as Eq. (28). As U and V are arbitrary functions of the dilaton field Φ, we can not get the exact form of the $O(\omega^2)$ term. Only when we choose a specific model we know the exact form of U and V then we can calculate the $O(\omega^2)$ term. Appendix A provides an example. Neglecting the $O(\omega^2)$ term and comparing with the Boltzmann factor again, we recover the radiation spectrum and the radiation temperature can be expressed as

$$T = \frac{w'}{4\pi}\,. \quad (37)$$

Taking the trace of Eq. (4) in the vacuum case gives $\Box\Phi = V$. Combining the equation (24) and the definition (28), we have that

$$T = \frac{\kappa}{2\pi}\,, \quad (38)$$

which agrees with the general result in the previous section.

As we can see, in the stationary case, the vector (10) is the same as the Kodama-like vector field and becomes the Killing vector field of the spacetime. When the spacetime is asymptotically flat the relative energy ω becomes the particle energy measured by the observers at infinity. So the radiation associated to the horizon is exactly the Hawking radiation and the temperature T is the temperature of the Hawking radiation.

5 Conclusions

In this paper, we used the Hamilton–Jacobi tunnelling method to analyze the tunnelling phenomenon in a general two-dimensional dilaton gravity theory first. We showed that the radiation exists in general spacetimes and can be associated to the apparent horizon and obtained the spectrum of this radiation. The temperature of the radiation was read out from this spectrum and it satisfied the usual relationship with the surface gravity. These are universal conclusions for all GDTs because all GDTs considered so far (e.g. the CGHS model) could be extracted from the action (1).[41] We also used the null geodesic method to analyze the stationary case of the theory and conformed our result. This suggests that, the tunnelling phenomenon is a universal phenomenon of the apparent horizon which is independent of the effects of "large isometries"[29] in the two-dimensional dilaton gravity. This is the same as the case in higher dimensions.

Acknowledgements

LMC would like to thank Rong-Gen Cai for his useful discussion and comments.

Appendix A: Tunnelling Phenomenon in the SRG Theory

For the spherically reduced gravity (SRG) model, the potential $U(\Phi)$ and $V(\Phi)$ are given respectively by

$$U(\Phi) = \frac{n-3}{n-2}\Phi^{-1}\,,$$
$$V(\Phi) = (n-2)(n-3)\lambda^2\Phi^{(n-4)/(n-2)}, \quad (\mathrm{A1})$$

where λ is a constant with dimension of mass square. When $n = 4$, the action (1) is

$$I = \int \mathrm{d}^2x\sqrt{-h}\Big[\Phi R + \frac{1}{2\Phi}D^a\Phi D_a\Phi + 2\lambda^2 + \mathscr{L}_m\Big]\,. \quad (\mathrm{A2})$$

With the definition (28), we can get

$$\Phi = \frac{r^2}{4}, \quad \mathrm{e}^{Q} = \frac{2}{r}, \quad w = 2\lambda^2 r\,. \quad (\mathrm{A3})$$

The vacuum solution (27) now is

$$\mathrm{d}s^2 = -\Big(4\lambda^2 - \frac{4m}{r}\Big)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r\,. \quad (\mathrm{A4})$$

In the null geodesic method, the imaginary part of the action of the outgoing particle is

$$\mathrm{ImS} = \int_m^{m-\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{\mathrm{d}r\mathrm{d}(m-\omega')}{1-\sqrt{1-[4\lambda^2-4(m-\omega')/r]}}$$
$$= -\frac{\pi}{2}\int_{r_{\rm in}}^{r_{\rm out}} r$$
$$= \frac{\pi\omega}{4\lambda^4}(2m-\omega)\,, \quad (\mathrm{A5})$$

where $r_{\rm in} = m/\lambda^2$ and $r_{\rm out} = (m-\omega)/\lambda^2$. To the first order of ω, we get the radiation temperature as

$$T = \frac{\lambda^4}{\pi m}\,, \quad (\mathrm{A6})$$

and then the surface gravity is

$$\kappa = \frac{2\lambda^4}{m}\,. \quad (\mathrm{A7})$$

With the metric (A4), we can get

$$\frac{1}{2}\,\mathrm{e}^{Q}\Box\Phi|_h = \frac{2\lambda^4}{m}\,, \quad (\mathrm{A8})$$

after some simple calculation. Note that we get the ω^2 term, which comes from the back-reaction of particle emission.

References

[1] S. W. Hawking, Nature (London) **248** (1974) 30.

[2] S. W. Hawking, Commun. Math. Phys. **43** (1975) 199 [167(1975)].

[3] R. Parentani and R. Brout, Int. J. Mod. Phys. D **1** (1992) 169.

[4] R. Brout, S. Massar, R. Parentani, and Ph. Spindel, Phys. Rept. **260** (1995) 329.

[5] Jennie H. Traschen, *An Introduction to Black Hole Evaporation*, In *Mathematical Methods in Physics*, Proceedings, Winter School, Londrina, Brazil, August 17–26, (1999).

[6] Ted Jacobson, *Introduction to Quantum Fields in Curved Space-Time and the Hawking Effect*, In *Lectures on Quantum Gravity, Proceedings, School of Quantum Gravity*, Valdivia, Chile, January 4–14, 2002, pp. 39–89 (2003).

[7] Don N. Page, New J. Phys. **7** (2005) 203.

[8] Maulik K. Parikh, Int. J. Mod. Phys. D **13** (2004) 2351, [Gen. Rel. Grav. **36** (2004) 2419].

[9] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15** (1977) 2752.

[10] Ran Li and Ji-Rong Ren, Class. Quant. Grav. **25** (2008) 125016.

[11] Abhay Ashtekar and Badri Krishnan, Living Rev. Rel. **7** (2004) 10.

[12] Petr Hajicek, Phys. Rev. D **36** (1987) 1065.

[13] Abhay Ashtekar and Martin Bojowald, Class. Quant. Grav. **22** (2005) 3349.

[14] Per Kraus and Frank Wilczek, Nucl. Phys. B **433** (1995) 403.

[15] Per Kraus and Frank Wilczek, Nucl. Phys. B **437** (1995) 231.

[16] Esko Keski-Vakkuri and Per Kraus, Phys. Rev. D **54** (1996) 7407.

[17] K. Srinivasan and T. Padmanabhan, Phys. Rev. D **60** (1999) 024007.

[18] S. Shankaranarayanan, T. Padmanabhan, and K. Srinivasan, Class. Quant. Grav. **19** (2002) 2671.

[19] Chikun Ding, Mengjie Wang, and Jiliang Jing, Phys. Lett. B **676** (2009) 99.

[20] Maulik K. Parikh and Frank Wilczek, Phys. Rev. Lett. **85** (2000) 5042.

[21] Marco Angheben, Mario Nadalini, Luciano Vanzo, and Sergio Zerbini, J. High Energy Phys. **05** (2005) 014.

[22] T. Padmanabhan, Phys. Rept. **406** (2005) 49.

[23] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini, and S. Zerbini, Class. Quant. Grav. **26** (2009) 062001.

[24] Rong-Gen Cai, Li-Ming Cao, and Ya-Peng Hu, Class. Quant. Grav. **26** (2009) 155018.

[25] Rabin Banerjee and Bibhas Ranjan Majhi, J. High Energy Phys. **06** (2008) 095.

[26] Matt Visser, Int. J. Mod. Phys. D **12** (2003) 649.

[27] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini, and G. Zoccatelli, Phys. Lett. B **657** (2007) 107.

[28] Ryan Kerner and Robert B. Mann, Phys. Rev. D **75** (2007) 084022.

[29] L. Vanzo, G. Acquaviva, and R. Di Criscienzo, Class. Quant. Grav. **28** (2011) 183001.

[30] Shu-Zheng Yang and De-You Chen, Int. J. Theor. Phys. **46** (2007) 1747.

[31] Xiao-Ning Wu and Si-Jie Gao, Phys. Rev. D **75** (2007) 044027.

[32] Timothy Clifton, Class. Quant. Grav. **25** (2008) 175022.

[33] Jiliang Jing and Yongjiu Wang, Int. J. Theor. Phys. **35** (1996) 1481.

[34] Ji-Liang Jing and Yong-Jiu Wang, Int. J. Theor. Phys. **36** (1997) 1745.

[35] Hideo Kodama, Prog. Theor. Phys. **63** (1980) 1217.

[36] Sean A. Hayward, Class. Quant. Grav. **15** (1998) 3147.

[37] Sean A. Hayward, Phys. Rev. Lett. **81** (1998) 4557.

[38] José M. M. Senovilla and Ramón Torres, Class. Quant. Grav. **32** (2015) 085004, [Erratum: Class. Quant. Grav. **32** (2015) 189501].

[39] Curtis G. Callan, Jr., Steven B. Giddings, Jeffrey A. Harvey, and Andrew Strominger, Phys. Rev. D **45** (1992) R1005.

[40] Shin'ichi Nojiri and Sergei D. Odintsov, Int. J. Mod. Phys. A **16** (2001) 1015.

[41] D. Grumiller, W. Kummer, and D. V. Vassilevich, Phys. Rept. **369** (2002) 327.

[42] Jose P. S. Lemos and Paulo M. Sa, Phys. Rev. D **49** (1994) 2897, [Erratum: Phys. Rev. D **51** (1995) 5967].

[43] Rong-Gen Cai and Li-Ming Cao, *Mechanics of Apparent Horizon in Two Dimensional Dilaton Gravity*, arXiv:1609.08306, 2016.

[44] P. Painlevé, C. R. Acad. Sci. (Paris) **173** (1921) 677.

[45] Maulik K. Parikh, *Energy Conservation and Hawking Radiation*, In *on Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories*, Proceedings, 10th Marcel Grossmann Meeting, MG10, Rio de Janeiro, Brazil, July 20-26, 2003. Pt. A-C (2004) pp. 1585–1590.