

Near horizon soft hairs as microstates of generic AdS₃ black holesM. M. Sheikh-Jabbari^{1,*} and H. Yavartanoo^{2,†}¹*School of Physics, Institute for Research in Fundamental Sciences (IPM),
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In [1] the *horizon fluffs* proposal is put forward to identify microstates of generic nonextremal three-dimensional Bañados–Teitelboim–Zanelli (BTZ) black holes. The proposal is that black hole microstates, the horizon fluffs, are certain near horizon soft hairs which are not in the coadjoint orbits of the asymptotic Virasoro algebra associated with the BTZ black holes. It is also known that AdS₃ Einstein gravity has more general black hole solutions than the BTZ family which are generically described by two periodic, but otherwise arbitrary, holomorphic and antiholomorphic functions. We show that these general AdS₃ black holes which are typically conformal descendants of the BTZ black holes and are characterized by the associated Virasoro coadjoint orbits, appear as coherent states in the asymptotic symmetry algebra corresponding to the black hole family. We apply the horizon fluffs proposal to these generic AdS₃ black holes and identify the corresponding microstates. We then perform microstate counting and compute the entropy. The entropy appears to be an orbit invariant quantity, providing an important check for the horizon fluffs proposal.

DOI: [10.1103/PhysRevD.95.044007](https://doi.org/10.1103/PhysRevD.95.044007)**I. INTRODUCTION**

Thermodynamic behavior of black holes plus requirement of unitary evolution for black hole formation and evaporation, hint to the presence of an underlying black hole microstate system. Black hole microstates are often attributed to a quantum theory of gravity, and to states which have no counterpart in the classical gravity theory. In particular and within string theory, following [2], there have been many efforts to identify certain supersymmetric black hole microstates, e.g., see [3]. The ideas from string theory seem not to be applicable to the simplest, realistic black holes like Schwarzschild or Kerr.

On the other hand, after the seminal works Bondi and Mentzer, and Sachs (BMS) [4], and Brown and Henneaux [5], we have learned that in any theory with local gauge symmetry, one may be able to define nonlocal conserved charges given through surface integrals. For example see [6–8] and references therein. These charges are associated to a certain (measure-zero) subset of local gauge transformations which remain undetermined even after a complete gauge fixing and may hence be dubbed as residual gauge symmetries [9]. These charges satisfy an algebra which is generically infinite dimensional and may admit central extension. It is then natural to assume that states in the quantum Hilbert space of theories with local gauge symmetries can also carry these nonlocal (residual gauge symmetry) charges. Therefore, there is an extended Hilbert

space which is a direct product of the usual gauge theory Hilbert space (e.g., containing transverse photon states in the case of QED or transverse traceless gravitons in the case of Einstein general relativity) and the states carrying additional quantum numbers associated with residual gauge symmetries [10]. The states which only differ in their residual gauge symmetry charges, as these charges are associated with gauge transformations, have the same energy and momentum. These charges may, hence, be also called “soft charges” [11].

In the context of general relativity we are dealing with diffeomorphisms as local gauge symmetries and the soft charges may be associated with the residual diffeomorphisms [9]. The residual diffeomorphisms are usually identified through prescribing appropriate fall off behavior for the “asymptotic” field (metric) perturbations caused by the diffeomorphisms. The BMS [4] and Brown-Henneaux [5] have shown the prime examples of such analyses. If one can identify the charges associated with residual diffeomorphisms and their symmetry algebra, then geometries which are diffeomorphic to each other but have different residual diffeomorphism charges, become physically distinguishable. One may hence explore the idea whether existence of these soft charges can remedy the pressing questions in the context of black hole physics, like black hole microstate problem or information paradox [11]. In this context one may hence call such states as “soft hair” [11], inspired by (and in contrast to) the “(no) hairs” statements of black hole [12].

Motivated by these ideas “asymptotic symmetry group” of many different background geometries have been

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studied, e.g., see [13,14], and the corresponding representations, which in this context is more appropriately called (coadjoint) orbits of the algebra, have been analyzed [15,16]. In the three dimensional (3d) asymptotic AdS₃ cases, this symmetry algebra consists of two copies of Virasoro algebra at the Brown-Henneaux central charge [5], whereas for 3d or 4d asymptotic flat geometries we are dealing with BMS₃ and BMS₄ algebras [14]. In this work we will focus on asymptotic locally AdS₃ geometries with flat 2d cylinder as the 2d boundary metric.

In black hole geometries we have a horizon and, near horizon region of a generic black hole geometry with bifurcate horizon has a 2d Rindler wedge. It has been noted that at the near horizon geometry one has the possibility of imposing nontrivial boundary conditions on the metric fluctuations leading to a near horizon symmetry algebra [17–21].¹ This symmetry algebra generically leads to a specific realization of the asymptotic symmetry algebra.² Despite the possibility of different boundary conditions, analysis of [21,22] suggests that for all different near horizon geometries there are always boundary conditions which lead to infinite number of copies of Heisenberg algebra. For the case of 3d geometries this is the algebra of creation-annihilation operators of a 2d free boson theory [21].

3d Einstein gravity has been viewed as a testbed for asking questions about quantum gravity and black holes, e.g., see [24]. In particular, after the seminal work of Bañados-Zanelli-Teitelboim (BTZ) [25], it is known that 3d Einstein gravity with negative cosmological constant (AdS₃ gravity) admits black hole solutions. In this work we will focus on AdS₃ gravity and consider a more general class of black holes (than the BTZ solution), those within the class of Bañados geometries [26] and analyze their near horizon and asymptotic symmetries, and identify their microstates.

In [1] a fairly simple proposal was made for identifying black hole microstates: There are soft hairs labeled by the near horizon symmetry algebra charges. There is a certain class of near horizon soft hairs which are not distinguishable by the asymptotic symmetry algebra charges associated with the nonextremal black hole solution or its conformal descendants. The proposal is that, a specific class of these states which were called *horizon fluffs*, are the black hole microstates. In this work we extend and generalize the analysis of [1] for nonextremal black holes in the class of Bañados geometries. This analysis, besides clarifying some aspects, provides a further nontrivial check of the black hole horizon fluffs proposal: as was discussed

¹There seems to be some different choices for the near horizon boundary conditions and hence different near horizon algebras. See [22] for a recent and detailed discussion on this issue.

²For the case of degenerate horizon of extremal black holes, instead of the Rindler wedge we get an AdS₂ geometry. The near horizon symmetry algebra in this case is different than the generic nonextreme black holes [23].

in some detail in [27–29] and as we will briefly review in the next section, Bañados geometries may be classified by the coadjoint orbits [30,31] of asymptotic Virasoro symmetry algebra. Then, as is discussed in [28] entropy should be an *orbit invariant* quantity. Identifying microstates for generic black holes in the Bañados family of solutions, we explicitly construct the horizon fluffs and confirm that the entropy is an orbit invariant.

This paper is organized as follows. In Sec. II, we review Bañados geometries and simple facts about them. This is essentially a review of [28,29]. In Sec. III, we present the near horizon algebra associated with generic AdS₃ black holes and its relation to the Brown-Henneaux asymptotic symmetry algebra. In Sec. IV, we construct the Hilbert space of the near horizon algebra. We show that the Bañados geometry is a coherent state in the asymptotic Hilbert space. We also review how the states in asymptotic and near horizon Hilbert spaces fall into the Virasoro coadjoint orbits. In Sec. V, we present the definition of black hole microstates through an equation over the near horizon Hilbert space. Solving this equation we identify the microstates and perform their counting. Section VI summarizes our analysis and results and presents an outlook.

II. A QUICK REVIEW ON BAÑADOS GEOMETRIES

Recalling the Lagrangian and equations of motion of AdS₃ Einstein gravity,

$$\mathcal{L} = \frac{1}{16\pi G} \left(R + \frac{2}{\ell^2} \right), \quad R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu}, \quad (2.1)$$

and that in 3d Ricci tensor completely determines Riemann curvature, the set of solutions to this theory are all locally AdS₃ geometries. These solutions are all locally diffeomorphic to each other and can become distinct solutions only through their residual diffeomorphism charges (if they exist). As is now established, thanks to the work of Brown and Henneaux [5], the set of distinct geometries may be fully specified by the boundary behavior of the geometries. The set of such geometries which obey the standard Brown-Henneaux boundary conditions [5] are given as [26]³

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \frac{\ell^2 L_-(x^-)}{r} dx^- \right) \times \left(r dx^- - \frac{\ell^2 L_+(x^+)}{r} dx^+ \right) \quad (2.2)$$

where $L_{\pm}(x^{\pm})$ are two arbitrary periodic functions $L_{\pm}(x^{\pm} + 2\pi) = L_{\pm}(x^{\pm})$. We will call the metrics in (2.2)

³There are more relaxed boundary conditions than the Brown-Henneaux ones, leading to geometries with more independent functions, e.g., see [32,33].

Bañados geometries. The conformal/causal boundary of these geometries is a cylinder parametrized by t, ϕ , where

$$x^\pm = t \pm \phi, \quad \phi \in [0, 2\pi].$$

Note that as discussed in [28] range of r coordinate may be extended to regions with negative r^2 .

The special case of constant, positive L_\pm constitute the BTZ black hole [25] family, while constant $-1/4 < L_\pm < 0$ family are the conic spaces and $L_\pm = -1/4$ is the global AdS_3 .

For the generic functions $L_\pm(x^\pm)$, being locally AdS_3 , Bañados geometries have six Killing vector fields forming an $sl(2, R) \times sl(2, R)$ algebra. These six Killing vector fields are of the form [27],

$$\chi[K_+, K_-] = \chi^r \partial_r + \chi^+ \partial_+ + \chi^- \partial_-, \quad (2.3a)$$

$$\begin{aligned} \chi^r &= -\frac{r}{2}(K'_+ + K'_-), \\ \chi^+ &= K_+ + \frac{\ell^2 r^2 K''_- + \ell^4 L_- K''_+}{2(r^4 - \ell^4 L_+ L_-)}, \\ \chi^- &= K_- + \frac{\ell^2 r^2 K''_+ + \ell^4 L_+ K''_-}{2(r^4 - \ell^4 L_+ L_-)}, \end{aligned} \quad (2.3b)$$

where primes denote derivative with respect to the argument and K_\pm are two functions obeying the third order differential equation

$$K'''_\pm - 4K'_\pm L_\pm - 2K_\pm L'_\pm = 0. \quad (2.4)$$

One may show that the solutions to the above equation can be specified by the solutions to the second order equations

$$\psi''_\pm - L_\pm \psi_\pm = 0. \quad (2.5)$$

The above equation with periodic functions L_\pm is called Hill's equation [34] and if we denote the two linearly independent solutions to the Hill's equation by $\psi_{\alpha\pm}$, $\alpha = 1, 2$ the three solutions to each equation in (2.4) are

$$K_\pm^0 = \psi_{1\pm} \psi_{2\pm}, \quad K_\pm^1 = \psi_{1\pm} \psi_{1\pm}, \quad K_\pm^{-1} = \psi_{2\pm} \psi_{2\pm}. \quad (2.6)$$

Out of these six K_\pm 's, recalling the Floquet theorem [34], the K_\pm^0 are always periodic and lead to two globally defined $U(1)$ Killing vector fields [27]. One may associate Komar-type conserved charges to the two Killing vectors [28,35].

Within the Bañados family, geometries for which $\frac{\psi'_{\alpha\sigma}}{\psi_{\alpha\sigma}}$ ($\sigma = \pm, \alpha = 1, 2$) are real-valued correspond to black

holes geometries with event and Killing horizons.⁴ BTZ family (with constant L_\pm) are very special cases in this class. The four branches of the bifurcate inner and outer Killing horizons are located at [28]

$$r_{\alpha\beta}^2 = \ell^2 \frac{\psi'_{\alpha+} \psi'_{\beta-}}{\psi_{\alpha+} \psi_{\beta-}}, \quad \alpha, \beta = 1, 2, \quad (2.7)$$

where two bifurcate horizons are the location of $r_{\alpha\beta}$ intersections. All the geometric properties of Bañados geometries may be conveniently described through ψ_\pm functions. More analysis and discussion on Bañados geometries may be found in [28,36] and references therein.

For the latter use we also note that (2.5) may be solved through the following ansatz:

$$\psi_\pm(x^\pm) = \exp\left(\frac{6}{c} \int^{x^\pm} J_\pm(x^\pm)\right) \quad (2.8)$$

where

$$J'_\pm + \frac{6}{c} J_\pm^2 = \frac{c}{6} L_\pm. \quad (2.9)$$

The above equation determines J_\pm for any given L_\pm and has two solutions $J_{\alpha\pm}$. For the family of black holes, hence, $J_\pm(x^\pm)$ are real-valued, periodic functions. This among other things, recalling (2.9), implies that $\int_0^{2\pi} L_\pm$ for black hole solutions are positive-definite, in accord with what is expected from the corresponding Virasoro coadjoint orbits [28,31]. The horizon radii (2.7) take a very simple form in terms of $J_{\alpha\pm}$ functions, $r_{\alpha\beta}^2 = \frac{36\ell^2}{c^2} J_{\alpha+} J_{\beta-}$. In other words, functions $J_{\alpha\pm}$ determine the ‘‘shape’’ of the horizon surfaces.

We note that the family of locally AdS_3 geometries discussed in [21] are specific family of Bañados geometries for which J_\pm are real valued and whose value of L_\pm are related to the γ, ω functions there through (2.9) where $J_\pm = \gamma \pm \omega$. We shall return to this point in our analysis in the next sections. This point will be further discussed and explored in an upcoming publication [37].

Finally, we would like to mention that, if the AdS_3 gravity theory (2.1) has a 2d CFT dual, this CFT should be at the Brown-Henneaux central charge c

$$c = \frac{3\ell}{2G}, \quad (2.10)$$

⁴Note that although one can always write two independent solutions of the Hill equation as real functions, however here by $\psi_{\alpha\sigma}$ we are referring to a particular form of solutions appearing in the Floquet theorem: $\psi_1(x) = e^{Tx} P_1(x), \psi_2(x) = e^{-Tx} P_2(x)$ where P_1, P_2 are two periodic smooth functions and T is the Floquet index (e.g., see [28]). The Floquet index T can be real or pure imaginary. The black hole family we will be discussing here corresponds to real Floquet index case.

and to have a well-defined semiclassical gravity description, $c \gg 1$. The Bañados geometries correspond to (set of) states in the 2d CFT for which the expectation value of the left and right sectors of the 2d energy momentum tensor \mathbf{T}_\pm are given as⁵

$$\langle \mathbf{T}_\pm \rangle = \frac{c}{6} L_\pm = \frac{6}{c} J_\pm^2 + J'_\pm. \quad (2.11)$$

III. THE TWO, NEAR HORIZON AND ASYMPTOTIC, SYMMETRY ALGEBRAS

As discussed in the introduction, depending on the choice of fall behavior of the metric perturbations near the boundary we can have residual diffeomorphisms which respect this fall off behavior. Besides the near boundary (asymptotic) behavior, for the geometries which have a horizon, one may explore the question whether we have a choice of boundary conditions which lead to nontrivial residual near horizon diffeomorphisms. If yes, we will then have (at least) two sets of residual diffeomorphisms and the corresponding charges and algebras. As we will see and discuss, however, despite being seemingly different, these two algebras and the corresponding Hilbert spaces are closely related to each. Explicitly, the near horizon Hilbert space is a specific subset of the asymptotic Hilbert space. In this section we review the two algebras and their relation. In the next section we will study the relation between the corresponding Hilbert spaces.

Note on conventions: Through this paper we use bold-face characters to denote operators and use calligraphic mathematical symbols to denote near horizon quantities. The near horizon algebra generators will hence be denoted as calligraphic-boldface. Expectation values of the operators would be denoted by the same symbol, but not boldfaced.

A. Near horizon algebra

As discussed one may get nontrivial symmetries through imposing appropriate boundary conditions on the near horizon geometry of a generic nonextreme black hole. In this case one can show that the near horizon algebra consists of copies of the creation-annihilation algebra [1,21,22]

$$[\mathcal{J}_n^\pm, \mathcal{J}_m^\pm] = \frac{1}{2} n \delta_{n,-m}, \quad [\mathcal{J}_n^+, \mathcal{J}_m^-] = 0. \quad (3.1)$$

The \mathcal{J}_n^\pm generators may be viewed as creation-annihilation operators for a free 2d boson theory on $\mathbb{R} \times S^1$ which

⁵We comment that L_\pm are real-valued periodic functions, (2.11), while implying J_\pm should be periodic, does not imply J_\pm should be real. In particular, $J_0^\pm = \frac{1}{2\pi} \int_0^{2\pi} J^\pm(x^\pm)$ can take pure imaginary or real values [37].

is a 2d conformal field theory (CFT₂) and the \mathcal{L}_n 's are Fourier modes of its energy-momentum tensor:

$$\mathcal{L}_n^\pm \equiv \sum_{p \in \mathbb{Z}} : \mathcal{J}_{n-p}^\pm \mathcal{J}_p^\pm : \quad (3.2)$$

where $::$ denotes normal ordering (\mathcal{J}_n with $n > 0$ may be viewed as annihilation operators). We then obtain

$$\begin{aligned} [\mathcal{L}_n^\pm, \mathcal{L}_m^\pm] &= (n-m) \mathcal{L}_{n+m}^\pm + \frac{1}{12} (n^3 - n) \delta_{n,-m}, \\ [\mathcal{L}_n^+, \mathcal{L}_m^-] &= 0. \end{aligned} \quad (3.3)$$

This is two copies of Virasoro algebra at central charge one plus a $u(1)$ current. Using (3.2), it would be also useful to find commutation relation between \mathcal{L}_n and \mathcal{J}_m . This turns out to be

$$[\mathcal{L}_n^\pm, \mathcal{J}_m^\pm] = -m \mathcal{J}_{n+m}^\pm, \quad [\mathcal{L}_n^+, \mathcal{J}_m^-] = 0. \quad (3.4)$$

We stress that, as the above explicitly shows, the $\mathcal{L}_n, \mathcal{J}_n$ generators are not independent and are related through (3.2). The ‘‘independent’’ part of the near horizon algebra is (3.1) and as we will see the Virasoro generators (3.2) which are constructed through the \mathcal{J}_n 's, are convenient operators for the comparison to the asymptotic symmetry algebra.

As some remarks, we note that (1) the near horizon algebra (3.1) is independent of the AdS₃ radius and the details of the black hole we started from; (2) \mathcal{J}_0^\pm are central elements in the algebra commuting with all the other generators; (3) as is seen from (3.2), the central charge of the near horizon Virasoro algebra, which is one, is purely quantum in nature. In the sense that it arises from the normal ordering in the definition (3.2).

B. Asymptotic Virasoro algebra

The asymptotic symmetry group of asymptotically AdS₃ geometries with the Brown-Henneaux boundary conditions has been analyzed in [5]; it is two copies of Virasoro algebra at the Brown-Henneaux central charge (2.10). Here we discuss a specific realisation of this algebra for black holes in the family of Bañados geometries.

1. Realization of asymptotic symmetry algebra for asymptotic AdS₃ black holes

In [21] it was noted that for black holes in the asymptotic AdS₃ geometries, the Brown-Henneaux boundary conditions instead of the AdS₃ boundary may also be imposed, in an appropriate coordinate system, near the horizon. Specifically, in [21] (see also [22]) a family of locally AdS₃ black holes which are specified by two arbitrary real-valued periodic functions was constructed and analyzed. As discussed in Sec. II, these black hole geometries are a subset of Bañados geometries and follow Brown-Henneaux

fall off behavior. These two functions were then argued to be associated with two current algebras, which expectedly, is (3.1). As discussed in [1] and will become clear in Sec. III C, we find it convenient to represent this algebra in a slightly different notation and use J_n^\pm to denote its generators [21]⁶:

$$[J_n^\pm, J_m^\pm] = \frac{c}{12} n \delta_{n,-m}, \quad [J_n^+, J_m^-] = 0, \quad (3.5)$$

where $n, m \in \mathbb{Z}$. The generators of the well-established Brown-Henneaux Virasoro algebra L_n is then naturally obtained from J_n through a twisted Sugawara construction

$$L_n^\pm \equiv i n J_n^\pm + \frac{6}{c} \sum_{p \in \mathbb{Z}} J_{n-p}^\pm J_p^\pm. \quad (3.6)$$

It is straightforward to check that

$$\begin{aligned} [L_n^\pm, L_m^\pm] &= (n-m) L_{n+m}^\pm + \frac{c}{12} n^3 \delta_{n,-m}, \\ [L_n^\pm, J_m^\pm] &= -m J_{n+m}^\pm + i \frac{c}{12} m^2 \delta_{n,-m}, \end{aligned} \quad (3.7)$$

and $[X^+, Y^-] = 0$ for any X, Y .⁷

It is notable that the above is nothing but (2.9) and (2.11) and that the Virasoro generators L_n are Fourier modes of the 2d energy momentum tensor T .⁸ In other words, the functions ψ_\pm are related to the expectation value of the Wilson line operator (of the $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ Chern-Simons formulation of AdS₃ gravity) made from the holomorphic current fields $J_\pm(x^\pm)$ [21]. Note also that

⁶We would like to thank H. Afshar and D. Grumiller for clarifying explanations on this issue.

⁷As mentioned our asymptotic algebra with generators L_n^\pm, J_n^\pm arises from the usual asymptotic Brown-Henneaux boundary conditions which allows for a holomorphic and an antiholomorphic functions. However, in the AdS₃ case one has the possibility of relaxing the Brown-Henneaux boundary conditions. For example, the boundary conditions allowing for boundary metric to vary up to a conformal transformation was considered in [32]. This relaxed boundary conditions leads to the enhanced (asymptotic) symmetry algebra [32] which is a $u(1)$ Kac-Moody algebra where now the Virasoro and current generators are independent and not related as in (3.6). Note that besides the $L(J)$ relation (3.6), the algebra in [32] is different than our algebra (3.7), as the $[L_n, J_m]$ commutator there do not involve the anomaly term proportional to m^2 . We should also note that there are other possibilities for boundary condition leading to other algebras, e.g., see [33,38].

⁸The twisted Sugawara construction is closely associated with a one-dimensional linear dilaton background string theory [21]. The string worldsheet field comes from the conformal factor of the 2d x^\pm part of the 3d metric. Intriguingly, the ‘‘string tension’’ α' of this theory is proportional to central charge c (in the semiclassical large c limit) and the linear dilaton background arises from the cosmological constant of the 3d geometry.

as (2.7) shows, the equation defining the bifurcate horizons are directly related to the product of J^\pm currents.⁹

J_0^\pm is a central element of the algebra and commutes with all J_n^\pm and L_n^\pm 's. This observation makes a direct connection with the surface charge analysis of [29,35], where it was argued that the charges associated with the two exact symmetries (the two periodic Killing vectors) should commute with the Virasoro generators. Explicitly, one may identify the two exact symmetry charges (which in [28,29] was denoted by J_\pm) as a function of the center element J_0^\pm , as we will discuss below $J_\pm \propto (J_0^\pm)^2$.

2. ‘‘Asymptotic’’ and ‘‘near horizon’’ symmetry algebras are defined everywhere

The usual way of identifying residual symmetries and the associated charge algebras is through imposing certain boundary conditions in the asymptotic region of spacetime. However, there are cases where one can provide alternative ways to identify the symmetries. Technically, and for the example of Bañados geometries, one can view this set of solutions as a phase space with a given symplectic two-form. Then, the conserved charge would be related to transformations on the phase space which do not change the symplectic two-form. In this case, the ‘‘asymptotic’’ symmetries become ‘‘symplectic’’ symmetries. For the case of Bañados geometries this has been demonstrated in [29]. For other cases of symplectic symmetries see [35,40,41].

Promoting the asymptotic symmetries to symplectic symmetries means that the conditions defining these charges (like ‘‘the fall off behavior’’) and the surface integrals, integrals over codimension two compact space-like surfaces, can be chosen or specified not only in the asymptotic region but at any place in spacetime. However, one should make sure that there are no topological obstructions, such as singularities, or physical obstructions, like closed-timelike curves (CTC's), in the spacetime. Dealing with symplectic symmetries means that the algebras (3.5) and (3.7) can be defined everywhere in the geometry, in the near horizon and the asymptotic regions.¹⁰ Moreover, one can show [39] that the algebra (3.1) or equivalently (3.5) is the symplectic symmetry of the family of black hole geometries discussed in [21,22]. Despite these facts and to distinguish this algebra from the one appearing on the near horizon, we keep calling them as ‘‘asymptotic’’ symmetries.

⁹We note that one may still enhance this algebra by the addition of two more generators Q^\pm which are ‘‘conjugate’’ to J_0^\pm , i.e. $[J_n^\pm, Q^\pm] = \frac{ic}{12} \delta_{n,0}$, $[L_n^\pm, Q^\pm] = i J_n^\pm$ [39]. The Q^\pm were also noted in Eq. (7.9) [32], where the Q generators there are related to Q^\pm as $Q = Q^+ + Q^-$.

¹⁰For the case of 3d gravity, where we have a Chern-Simons description, the fact that charges may be defined at any radius r , and are r -independent can be seen very explicitly from the fact that r -dependence, in the Chern-Simons perturbations can be removed by a gauge choice, e.g., see [19,21,33,42–44].

C. Asymptotic algebra from near horizon algebra

As discussed the near horizon algebra (3.1) or (3.5), and the Brown-Henneaux Virasoro algebra (3.7) are both symplectic symmetries of the phase space of locally AdS₃ black hole solutions and may hence be defined everywhere in the corresponding coordinate systems [21,29,39]. Moreover, from \mathcal{J}_n^\pm one construct \mathcal{L}_n^\pm (3.2) which generate a Virasoro algebra at central charge one.

In [45] there is a simple proposal how to relate two Virasoro algebras at central charge c and at central charge one. Motivated by the proposal in [45], we related the associated $L_0^\pm, \mathcal{L}_0^\pm$ as

$$cL_0^\pm = \mathcal{L}_0^\pm - \frac{1}{24}. \quad (3.8)$$

The extra $-1/24$ term in the right-hand side (RHS) has appeared because of difference between the two conventions used in the ‘‘near horizon’’ (3.2) and ‘‘asymptotic’’ (3.7) Virasoro algebras. The relation (3.8), while could be (partially) justified with the arguments in [45], is a part of our horizon fluffs proposal.

We next require that the relation (3.8) is consistent with the commutation relations among other generators. This requirement then yields [1]¹¹

$$J_n^\pm = \frac{1}{\sqrt{6}} \mathcal{J}_{cn}^\pm, \quad n \neq 0. \quad (3.9)$$

In the above we have assumed central charge c is integer-valued. This assumption is a well justified one, noting the discussions of [46] and that the modular invariance of a presumed unitary 2d CFT dual to the pure AdS₃ Einstein gravity implies the central charge c to be a multiple of 24.¹² Equation (3.9) does not relate \mathcal{J}_0^\pm and J_0^\pm . The relation between these two is induced from (3.8):

¹¹Recalling that the two algebras (3.1) and (3.5) are the same, up to possibly normalization of charges which is not fixed through the usual surface charge computation methods, we get a relation of the form

$$J_n^\pm = \sqrt{\frac{c}{6x}} \mathcal{J}_{xn}^\pm, \quad n \neq 0,$$

where x can be any arbitrary integer. The $x = c$ choice is fixed upon imposing (3.8).

¹²Analysis of [46] casts some doubts on having a well-defined pure AdS₃ *quantum* gravity (which is dual to a unitary-modular invariant 2d CFT). Nonetheless, we are working in the large c regime where there is a semiclassical gravity description and integer c assumption is a justified one in this limit.

$$\begin{aligned} L_0^\pm &= \frac{1}{c} \left(\mathcal{L}_0^\pm - \frac{1}{24} \right) \Rightarrow 6(J_0^\pm)^2 \\ &= (\mathcal{J}_0^\pm)^2 + \sum_{n \neq 0} \left(: \mathcal{J}_{-n}^\pm \mathcal{J}_n^\pm : - \sum_{n \neq 0} \mathcal{J}_{-nc}^\pm \mathcal{J}_{nc}^\pm \right) - \frac{1}{24}, \end{aligned} \quad (3.10)$$

where we used (3.6) and (3.2). Given the above one can find other L_n^\pm in terms of the near horizon algebra generators \mathcal{J}_n^\pm .

IV. HILBERT SPACE AND REPRESENTATIONS OF THE TWO ALGEBRAS

As already mentioned our symmetry algebras (3.1) and (3.7) are respectively (symplectic) symmetries of the family of metrics discussed in [21] and (2.2). This family of solutions may hence be viewed as representations or coadjoint orbits of the associated algebras. In a more quantum language, one may view them as (a part of) Hilbert space of the associated physical theory which is invariant under the corresponding algebra. Besides this geometric picture, we can directly construct the Hilbert space of the two algebras and study their relation. That is what we carry out in this section.

A. Asymptotic black hole Hilbert space \mathcal{H}_{BH} and black holes in Bañados geometries

The asymptotic algebra involves a Virasoro algebra at Brown-Henneaux central charge and hence Virasoro coadjoint orbits may be used to label Bañados geometries. This has been studied in some detail in [28,29] and we do not review that here. What we would like to note here is that realization of Virasoro generators in terms of $u(1)$ currents J_n^\pm , i.e. the twisted-Sugawara map (3.6), implies that only a subset of Virasoro coadjoint orbits associated with AdS₃ black holes, namely the hyperbolic orbits (see [28,31]) appear as coadjoint orbits of our L_n^\pm, J_n^\pm algebra.

To see the above explicitly, and for our later use, we construct these Virasoro orbits using the current algebra. To this end, we note that $J_n, n < 0$ and $J_n, n > 0$ are respectively creation annihilation conjugates of each other. Therefore, it is natural to define the ‘‘asymptotic black hole vacua’’ $|0; J_0^\pm\rangle_{BH}$ as

$$\begin{aligned} J_n^\pm |0; J_0^\pm\rangle_{BH} &= 0, \quad \forall n > 0, \\ J_0^\pm |0; J_0^\pm\rangle_{BH} &= J_0^\pm |0; J_0^\pm\rangle_{BH}, \end{aligned} \quad (4.1)$$

and we may choose the normalization such that

$${}_{BH}\langle 0; J_0^\pm | 0; J_0^\pm \rangle_{BH} = \delta_{J_0^\pm, J_0^\pm}. \quad (4.2)$$

Analysis of [21] show that J_0^\pm for set of AdS₃ black hole geometries discussed there should be real valued, which

without loss of generality may be chosen to take only positive values, i.e. $J_0^\pm \in \mathbb{R}^+$. This latter implies that J_0^\pm is self-adjoint (Hermitian) on the Hilbert space associated with this set of geometries. However, if we ignore the analysis of [21] and only focus on the algebra (3.5) or (3.6), there is no hermiticity condition induced on J_0^\pm and J_0^\pm can take real or imaginary values. We will return to this point, shortly in Sec. IV A 1. We define the asymptotic Hilbert space $\mathcal{H}_{\mathcal{BH}}$ as the states constructed upon vacuum states $|0; J_0^\pm\rangle_{\mathcal{BH}}$ with real-positive J_0^\pm .

The rest of states in the asymptotic Hilbert space $\mathcal{H}_{\mathcal{BH}}$ are then constructed by the action of $J_{-n}^\pm, n > 0$ on the vacuum. That is, $\mathcal{H}_{\mathcal{BH}}$ consists of states of the form

$$\begin{aligned} |\mathbf{J}(\{n_i^\pm\}); J_0^\pm\rangle_{\mathcal{BH}} &= \mathbf{J}(\{n_i^\pm\})|0; J_0^\pm\rangle_{\mathcal{BH}}, \\ \forall |\mathbf{J}(\{n_i^\pm\}); J_0^\pm\rangle_{\mathcal{BH}} &\in \mathcal{H}_{\mathcal{BH}}, \\ \mathbf{J}(\{n_i^\pm\}) &= \mathcal{N}_{\mathbf{J}} \prod_{\{n_i^\pm > 0\}} (J_{-n_i^\pm}^+ \cdot J_{-n_i^\pm}^-), \end{aligned} \quad (4.3)$$

where the normalization factor $\mathcal{N}_{\mathbf{J}}$ is fixed such that

$${}_{\mathcal{BH}}\langle \mathbf{J}(\{n_i'^\pm\}); J_0'^\pm | \mathbf{J}(\{n_i^\pm\}); J_0^\pm \rangle_{\mathcal{BH}} = \delta_{\{n_i'\}, \{n_i\}} \delta_{J_0'^\pm, J_0^\pm}, \quad (4.4)$$

where $\delta_{\{n_i'\}, \{n_i\}}$ is the product of all δ_{n_i', n_i} 's.

1. $\mathcal{H}_{\mathcal{BH}}$ vs. Hilbert space of unitary Virasoro (coadjoint) orbits $\mathcal{H}_{\mathcal{Vir}}$

Given a generic state, the associated Virasoro orbit may be constructed by the action of all possible products of $L_{-n}, n > 0$ on this state. Since we know the commutation of L_n, J_m we can readily distinguish Virasoro orbits in $\mathcal{H}_{\mathcal{BH}}$. In particular, since

$$[J_0^\pm, L_n^\pm] = 0, \quad (4.5)$$

all states in the same orbit have the same J_0^\pm . In other words, the orbits may be labeled by J_0^\pm , or J_0^\pm is an *orbit invariant quantity*.¹³ That is, there is an orbit associated with each $|0; J_0^\pm\rangle_{\mathcal{BH}}$ state, or, $|0; J_0^\pm\rangle_{\mathcal{BH}}$ is the representative of an orbit [31]. Recalling (3.6), one observes that

$$L_0^\pm |0; J_0^\pm\rangle_{\mathcal{BH}} \equiv L_0^\pm |0; J_0^\pm\rangle_{\mathcal{BH}} = \frac{6}{c} (J_0^\pm)^2 |0; J_0^\pm\rangle_{\mathcal{BH}}. \quad (4.6)$$

The condition for unitarity of the representation implies hermiticity of L_0^\pm and $L_0^\pm + \frac{c}{24} \geq 0$. In other words, if we

¹³We comment that $J^\pm(x^\pm) = J_0^\pm + \sum_{n \neq 0} J_n^\pm e^{inx^\pm}$ and (2.8) imply $\psi^\pm = \exp(\frac{6}{c} J_0^\pm x^\pm) P^\pm(x^\pm)$, where P^\pm are periodic functions. In other words, $\frac{6}{c} J_0^\pm$ are equal to the Floquet index, denoted by T_\pm in [28] (cf. footnote 4).

only focus on the Virasoro part of the algebra, one may relax hermiticity of J_0^\pm and require hermiticity of $(J_0^\pm)^2$ and $(J_0^\pm)^2 + (\frac{c}{12})^2 \geq 0$.

Recalling discussions in Secs. 3, 4 of [28], that the Bañados geometries are in one-to-one relation with the Virasoro coadjoint orbits and that the orbits discussed here are in one-to-one relation with the coadjoint orbits, the above implies that $\mathcal{H}_{\mathcal{BH}}$ which only includes real-valued J_0^\pm only captures hyperbolic orbits associated with BTZ black holes. If we allow for negative values of $(J_0^\pm)^2$, those with $-(\frac{c}{12})^2 < (J_0^\pm)^2 < 0$ capture elliptic orbits, associated with conic singularities on global AdS₃. The ‘‘global AdS₃ vacuum’’ has $(J_0^\pm)^2 = -(\frac{c}{12})^2$ [28].

Therefore, the set of all unitary representations of Virasoro algebra and their coadjoint orbits form a bigger Hilbert space than $\mathcal{H}_{\mathcal{BH}}$. This Hilbert space will conveniently be denoted by $\mathcal{H}_{\mathcal{Vir}}$ and contains orbits associated with global AdS₃ ($|0; J_0^\pm = ic/12\rangle$), elliptic orbits associated with ($|0; J_0^\pm = ic\nu^\pm/12\rangle$), $0 < \nu^\pm < 1$, and hyperbolic orbits associated with ($|0; J_0^\pm\rangle_{\mathcal{BH}}$), $J_0^\pm \in \mathbb{R}^+$ (BTZ black hole orbits). Therefore, it is readily seen that $\mathcal{H}_{\mathcal{BH}} \subset \mathcal{H}_{\mathcal{Vir}}$ [28,37].

Before closing this part some further comments are in order:

- (i) All states in the same Virasoro orbit will have the same J_0^\pm while they will have different L_0^\pm eigenvalues L_0^\pm . Explicitly, L_0^\pm is *not* an orbit invariant quantity.
- (ii) States in the orbit of $|0; J_0^\pm\rangle_{\mathcal{BH}}$ all have higher L_0^\pm eigenvalue than $|0; J_0^\pm\rangle_{\mathcal{BH}}$ itself. That is, the representative (which may be called ‘‘highest weight state’’) has the lowest value of L_0^\pm in the orbit.
- (iii) There could be more than one orbit associated with a given J_0^\pm , as the orbits can have another *discrete* label. This other label, being discrete, is not captured by the surface integrals accounting for charges associated with continuous transformations, and hence not included in our current discussions. These other orbits, however, are not among the representations which are unitary on a single cover of AdS₃, while they may be unitarizable on multiple covers of AdS₃. (See [28] for some preliminary discussion on the latter.) These other cases may be studied in a separate publication.

2. Generic black hole is a coherent state in $\mathcal{H}_{\mathcal{BH}}$

As mentioned in the previous subsections, the asymptotic Hilbert space $\mathcal{H}_{\mathcal{BH}}$ contains states with $J_0^\pm \in \mathbb{R}^+$ associated with geometries in the orbit of BTZ black holes (related to hyperbolic orbits). Geometries in these orbits are specified by generic functions $L_\pm(x^\pm)$ which can be mapped to constant-positive L_0^\pm , explicitly, there exists $h_\pm(x^\pm)$ functions such that [28]

$$L_{\pm}(x^{\pm}) = h'_{\pm}{}^2 L_0^{\pm} + S[h_{\pm}; x],$$

$$h_{\pm}(x^{\pm} + 2\pi) = h_{\pm}(x^{\pm}) + 2\pi,$$

where $S[h; x]$ is the Schwarz derivative.¹⁴ Alternatively, as reviewed in Sec. II (see also [28,36]) one may use ψ_{\pm} , or equivalently $J_{\pm}(x^{\pm})$, to specify the corresponding geometry. Explicitly, let us consider a geometry corresponding to the (coadjoint) orbit associated with the BTZ geometry with given J_0^{\pm} . This geometry is then associated to a state in \mathcal{H}_{BH} which we conveniently denote by $|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}}$. This state is defined through equation

$${}_{\text{BH}}\langle\{J_{n_i}^{\pm}\}; J_0^{\pm}|J_{n_k}^{\pm}|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}} = J_{n_k}^{\pm},$$

$$|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}} \in \mathcal{H}_{\text{BH}}, \quad (4.7)$$

where the $\{J_{n_i}^{\pm}\}$ appearing in $|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}}$ is the collection of all $J_{n_k}^{\pm}$ appearing in the right-hand side. The BTZ black hole discussed in [1] corresponds to $J_n^{\pm} = 0$, ($n \neq 0$) in the above family. Since in the BTZ orbit we are dealing with geometries with real $J_{\pm}(x^{\pm})$ functions, we have $J_{-n} = J_n^*$.

Recalling the fact that each J_n behaves like creation ($n < 0$) or annihilation ($n > 0$) operator, solutions to (4.7) are coherent states. Next, we also note that J_n, J_m commute for $n \neq -m$ and hence a general solution to set of equations (4.7) for all n is simply product of these coherent states. That is,

$$|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}} = \prod_{\{n_i^{\pm} > 0\}} \exp\left(\frac{12J_{n_i}^+ J_{-n_i}^+ - 12J_{n_i}^{+*} J_{n_i}^+}{cn_i}\right)$$

$$\cdot \exp\left(\frac{12J_{n_i}^- J_{-n_i}^- - 12J_{n_i}^{-*} J_{n_i}^-}{cn_i}\right) |0; J_0^{\pm}\rangle_{\text{BH}}. \quad (4.8)$$

One may readily check that the above coherent state is normalized ${}_{\text{BH}}\langle\{J_{n_i}^{\pm}\}; J_0^{\pm}|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}} = 1$.

As we see the black hole state is not a coherent state in J_0^{\pm} sector. This is due to the fact that J_0^{\pm} , unlike J_n^{\pm}, J_{-n}^{\pm} , have no conjugate in the algebra (3.5). However, as pointed out in footnote 9 one may enhance (3.5) by the addition of Q^{\pm} operators which are conjugate to J_0^{\pm} . We will briefly comment on this point and its possible implications for our black hole microstate counting in the discussion section.

Finally, we remark that using (3.6) one can read the values of L_n^{\pm} for the BTZ family in terms of their J_n^{\pm} eigenvalues, namely we have

$${}_{\text{BH}}\langle\{J_{n_i}^{\pm}\}; J_0^{\pm}|L_n^{\pm}|\{J_{n_i}^{\pm}\}; J_0^{\pm}\rangle_{\text{BH}} = L_n^{\pm}, \quad (4.9)$$

¹⁴Recalling (2.8), one may see that under a conformal transformation $J(x)$ behaves as: $J(x) \rightarrow J(h(x))h' - \frac{c}{12} \frac{h''}{h'}$, compatible with L, J commutator in (3.7).

where $L_n^{\pm} = inJ_n^{\pm} + \frac{6}{c} \sum J_{n-p}^{\pm} J_p^{\pm}$. As discussed values of L_n^{\pm} and in particular L_0^{\pm} are not orbit invariant quantities and the appropriate definition of mass or angular momentum, as inspired by the gravity analysis, is through orbit invariant quantities J_0^{\pm} . This latter leads to the conserved quantities associated with the two $U(1)$ global Killing vectors of Bañados geometries and are those appearing in the expression for the first law of thermodynamics [28,35]. For the special case of BTZ black hole, where all $J_n^{\pm}, n \neq 0$ vanish, then (cf. (2.11), (3.6))

$$\text{For BTZ geometry: } \frac{1}{2}(\ell M \pm J) \equiv \Delta^{\pm} = \frac{6}{c}(J_0^{\pm})^2. \quad (4.10)$$

where M, J are ADM mass and angular momentum of the BTZ black hole.

B. Hilbert space of near horizon soft hairs $\mathcal{H}_{\mathcal{NH}}$

As discussed in [28] the causal structure of all Bañados geometries in the same Virasoro (coadjoint) orbit are the same. Therefore, despite the fact that in the BTZ orbits we are dealing with geometries with nonconstant value of $L_{\pm}(x^{\pm})$ (i.e. with nonzero L_n^{\pm} values), they have the same causal structure as the corresponding BTZ. One may hence study their near horizon algebra as was done for BTZ black hole itself. Analysis of [1,21] readily goes through and one obtains the same asymptotic and near horizon algebras (3.7) and (3.1) for all members in the BTZ orbits.

To construct the Hilbert space of the near horizon algebra $\mathcal{H}_{\mathcal{NH}}$, we follow essentially the same steps as the previous subsection. We start with defining the vacuum $|0\rangle$:

$$\mathcal{J}_n^{\pm}|0\rangle = 0, \quad \forall n \geq 0. \quad (4.11)$$

We note the difference between the above and (4.1) and that here we set \mathcal{J}_0^{\pm} to have zero eigenvalue on the vacuum. In principle we could have chosen the vacuum to have a nonzero \mathcal{J}_0^{\pm} value. $\mathcal{J}_0^{\pm} = 0$ is a convenient choice as $\mathcal{L}_0^{\pm}|0\rangle = 0$. More importantly, it is a convenient choice because the energy in the near horizon observer frame is proportional to $\mathcal{J}_0^+ + \mathcal{J}_0^-$ [21]. With this choice, all the states in $\mathcal{H}_{\mathcal{NH}}$ will have zero energy from a near horizon observer viewpoint, they are *near horizon soft hairs* [1,11]. In other words, $\mathcal{H}_{\mathcal{NH}}$ is nothing but the Hilbert space of near horizon soft hairs.

A generic descendant of the vacuum, $|\Psi(\{n_i^{\pm}\})\rangle$, may then be constructed using creation operators $\mathcal{J}_{-n_i^{\pm}}^{\pm}$ with sets of positive integers $\{n_i^{\pm} > 0\}$, i.e.

$$|\Psi(\{n_i^{\pm}\})\rangle = \prod_{\{n_i^{\pm} > 0\}} (\mathcal{J}_{-n_i^+}^+ \cdot \mathcal{J}_{-n_i^-}^-)|0\rangle,$$

$$\forall |\Psi(\{n_i^{\pm}\})\rangle \in \mathcal{H}_{\mathcal{NH}}. \quad (4.12)$$

Given (4.11), we deduce that

$$\mathcal{L}_0^\pm |\Psi(\{n_i^\pm\})\rangle = \left(\sum_i n_i^\pm \right) |\Psi(\{n_i^\pm\})\rangle. \quad (4.13)$$

One may compute the eigenvalue of \mathcal{L}_n 's ($n \neq 0$) and construct the near horizon (Virasoro) orbits. However, since it is essentially the same as the asymptotic algebra case and will not be relevant for our microstate counting, we will not present it here.

C. Relation between $\mathcal{H}_{\mathcal{BH}}$, \mathcal{H}_{Vir} and $\mathcal{H}_{\mathcal{NH}}$

So far, we have introduced and constructed three Hilbert spaces:

- (i) $\mathcal{H}_{\mathcal{BH}}$ which is the Hilbert space of generic AdS₃ black holes, this includes all nonextremal BTZ black holes and their Virasoro excitations (descendants).
- (ii) \mathcal{H}_{Vir} which is the Hilbert space associated with all unitary constant representative Virasoro coadjoint orbits, including BTZ black holes and conic singularities and their Virasoro descendants. Every state in \mathcal{H}_{Vir} is then specified by the value of J_0^\pm and a set of integers (accounting for Virasoro excitations). Note in particular that $\mathcal{H}_{\mathcal{BH}} \subset \mathcal{H}_{\text{Vir}}$.
- (iii) $\mathcal{H}_{\mathcal{NH}}$ which denotes Hilbert space of near horizon soft hairs and its states are specified by a set of integers.

With the maps introduced in Sec. III C, we related a subset of the near horizon generators \mathcal{J}_n^\pm to the asymptotic ones J_n^\pm . Here we would like to discuss the relation between the Hilbert spaces induced from this mapping of generators. In particular, (3.9) and (3.8) or (3.10) may be viewed as an equation which could be written on the $\mathcal{H}_{\mathcal{NH}}$. Using them we learn that the $|0; J_0^\pm\rangle_{\mathcal{BH}}$ vacuum state is mapped onto several different states created by \mathcal{J}_r^\pm or product thereof, where r is *not* a multiple of c and $6(J_0^\pm)^2 = \sum r_i^\pm - 1/24$. Therefore, we learn that $\mathcal{H}_{\mathcal{NH}}$ contains all the states in $\mathcal{H}_{\mathcal{BH}}$ whose $6(J_0^\pm)^2$ is of the form given here and $\mathcal{H}_{\mathcal{NH}}$ does not include states with other form of J_0^\pm 's. These are the states which in [1] were identified as BTZ black hole microstates. See the next section for more discussions.

On the other hand, from discussions of Sec. III B 2 and the standard AdS₃/CFT₂ and Brown-Henneaux analysis, we expect $\mathcal{H}_{\mathcal{NH}} \subset \mathcal{H}_{\text{Vir}}$. This expectation will be discussed and established in [37]. However, here we would like to give the general picture. Given (3.9), one may distinguish two classes of creation operators among \mathcal{J}^\pm 's: \mathcal{J}_{-nc}^\pm with $n > 0$, and $\mathcal{J}_{-(nc+r)}^\pm$, $n > 0$, $r = 1, 2, \dots, c-1$. The first class, as discussed, are those which create states in the $\mathcal{H}_{\mathcal{BH}}$. The second class, which correspond to states with imaginary J_0^\pm in \mathcal{H}_{Vir} are then associated with certain conic defects and their Virasoro descendants. In particular, the second class contains conic defects with $L_0 = -1 + r/c$ where $r = 0$ case creates global AdS₃ vacuum and $r = c$

the massless BTZ state. These states are then creating the spectral flow between the two vacua. This picture dovetails with the well-known spectral flow of the (supersymmetric) 2d CFT's which are dual to AdS₃ gravity theory [47,48].

V. MICROSTATES OF A GENERIC AdS₃ BLACK HOLE

The most natural proposal for black hole microstates, which was put forward in [1], is that microstates of a black hole specified by a given set of L_n^\pm (or equivalently J_n^\pm) charges are states in $\mathcal{H}_{\mathcal{NH}}$ which describe a geometry with the same asymptotic charges L_n^\pm . To this end, we note that using the map (3.9) and also definitions of near horizon and asymptotic vacua, generators of the asymptotic algebra and in particular J_n^\pm can now be viewed as operators defined on the whole $\mathcal{H}_{\mathcal{NH}}$ (and not just on $\mathcal{H}_{\mathcal{BH}}$). Then, our proposed definition of black hole microstates associated with the Bañados geometry given by function $J_\pm(x^\pm)$ with Fourier modes J_n^\pm is simply equation (4.7), but now this equation is to be solved over $\mathcal{H}_{\mathcal{NH}}$. Explicitly, our microstates $|\mathcal{B}(\{J_n^\pm\}; J_0^\pm)\rangle \in \mathcal{H}_{\mathcal{NH}}$ are solutions to

$$\begin{aligned} \langle \mathcal{B}'(\{J_n^\pm\}; J_0^\pm) | J_n^\pm | \mathcal{B}(\{J_n^\pm\}; J_0^\pm) \rangle &= \frac{1}{\sqrt{6}} \\ \langle \mathcal{B}'(\{J_n^\pm\}; J_0^\pm) | \mathcal{J}_{nc}^\pm | \mathcal{B}(\{J_n^\pm\}; J_0^\pm) \rangle &= J_n^\pm \delta_{\mathcal{B}, \mathcal{B}'}, n \neq 0. \end{aligned} \quad (5.1)$$

Recalling the relations between asymptotic and near horizon generators (3.9) and that \mathcal{J}_n , J_m with n not a multiple of c , commute, solution to the above is of the form,

$$|\mathcal{B}(\{J_n^\pm\}; J_0^\pm)\rangle = \mathcal{N}_{\{n_i^\pm\}} |\mathcal{F}(\{n_i^\pm\}; J_0^\pm) \otimes \{|J_{n_i}^\pm\rangle_{\mathcal{BH}}\rangle \quad (5.2)$$

where

$$|\mathcal{N}_{\{n_i^\pm\}}|^2 = \prod_{\{n_i^\pm \text{ not a multiple of } c\}} \frac{2}{n_i^\pm} \cdot \frac{2}{n_i^\pm},$$

and

$$\begin{aligned} |\mathcal{F}(\{n_i^\pm\}; J_0^\pm)\rangle &= \prod_{\{n_i^\pm \text{ not a multiple of } c\}} (\mathcal{J}_{-n_i^+}^+ \cdot \mathcal{J}_{-n_i^-}^-) |0\rangle, \\ \text{such that } \sum n_i^\pm &= 6(J_0^\pm)^2 + \frac{1}{24}, \end{aligned} \quad (5.3)$$

or linear combinations thereof. In fact $|\mathcal{F}(\{n_i^\pm\})\rangle$ is exactly the microstates (horizon fluffs) of the BTZ black hole [1]. Hereafter, we will denote the set of states in (5.3) by $\mathcal{H}_{\mathcal{F}}$. It is readily seen that $\mathcal{H}_{\mathcal{F}}$ forms a vector space and that $\mathcal{H}_{\mathcal{BH}}$ and $\mathcal{H}_{\mathcal{F}}$ form two subspaces of $\mathcal{H}_{\mathcal{NH}}$ and overlap only on the vacuum state, as depicted in Fig. 1.

One may associate an orbit invariant mass and angular momentum to the BTZ orbits. This is possible recalling (2.11) and (3.6):

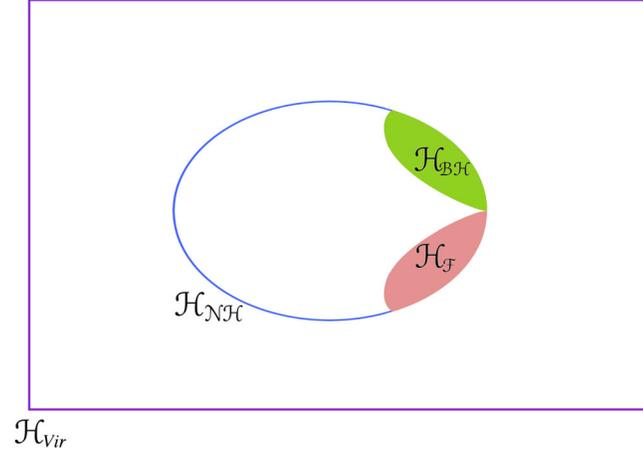


FIG. 1. A schematic diagram of the near horizon Hilbert spaces $\mathcal{H}_{\mathcal{NH}}$, Asymptotic black hole Hilbert space is $\mathcal{H}_{\mathcal{BH}}$ (green region) and the Hilbert space of horizon fluffs (black hole microstates) is $\mathcal{H}_{\mathcal{F}}$ (red region). Intersection of $\mathcal{H}_{\mathcal{BH}}$ and $\mathcal{H}_{\mathcal{F}}$ is the vacuum state $|0\rangle$ or $|0\rangle_{\mathcal{BH}}$

$$\begin{aligned} \Delta^\pm &= \frac{1}{2}(\ell M \pm J) = \frac{c}{6} \langle \mathcal{F}(\{n_i^\pm\}); J_0^\pm | \mathbf{L}_0^\pm | \mathcal{F}(\{n_i^\pm\}); J_0^\pm \rangle \\ &= \frac{6}{c} (J_0^\pm)^2 \simeq \frac{1}{c} \sum n_i, \end{aligned} \quad (5.4)$$

where in the last equality we assumed $\sum n_i \gg 1/24$ and dropped the $1/24$ part. Note that here n_i are not a multiple of c , that in the above the ‘‘BTZ part’’ of the black hole microstates, $|\mathcal{F}(\{n_i^\pm\}); J_0^\pm\rangle$ (and not $|\mathcal{B}(\{J_n^\pm\}; J_0^\pm)\rangle$) has been used, and that $\langle \mathcal{F}(\{n_i^\pm\}); J_0^\pm | \mathbf{J}_{n \neq 0}^\pm | \mathcal{F}(\{n_i^\pm\}); J_0^\pm \rangle = 0$.

A. Black hole microstate counting

Given the black hole microstates (5.2) corresponding to a generic geometry in the BTZ-orbit, we can perform their counting for a given asymptotic orbit invariant mass and angular momentum (5.4). The idea, as spelled out in [1], is to use Hardy–Ramanujan formula, see e.g., [49] and Refs. therein: number of partitions of a given positive integer N into non-negative integers, $p(N)$, for large N is given by

$$p(N)|_{N \gg 1} \simeq \frac{1}{4N\sqrt{3}} \exp\left(2\pi\sqrt{\frac{N}{6}}\right). \quad (5.5)$$

The number of microstates of any black hole in the BTZ orbit, recalling (5.4), is given by $p(c\Delta^+) \cdot p(c\Delta^-)$ and the entropy S by

$$\begin{aligned} S &= \ln p(c\Delta^+) + \ln p(c\Delta^-) \\ &= 2\pi \left(\sqrt{\frac{c\Delta^+}{6}} + \sqrt{\frac{c\Delta^-}{6}} \right) - \ln(c\Delta^+) - \ln(c\Delta^-) + \dots \end{aligned} \quad (5.6)$$

where we have assumed $c\Delta^\pm \gg 1$, and \dots denotes sub-leading terms in $c\Delta^\pm$.

Given the fact that a generic microstate of a black hole in the BTZ orbit is a direct product of a ‘‘microstate part’’ and an asymptotic ‘‘macrostate part’’ *cf.* (5.2), one readily sees that the microstate part is an orbit invariant quantity and only knows about orbit invariant (asymptotic) charges J_0^\pm . Therefore, the entropy is also an orbit invariant quantity. This is of course expected recalling discussions of [28] and that entropy is the conserved charge associated with an exact Killing vector field which generates the Killing horizon [28,35].

VI. DISCUSSIONS AND OUTLOOK

In this work we have repeated the analysis of the ‘‘horizon fluffs’’ proposal¹⁵ of [1] for generic black holes in the class of Bañados geometries. The ‘‘horizon fluffs’’ proposal has three main steps:

- (1) Identifying the given black hole as a state in the Hilbert space of asymptotic symmetry algebra $\mathcal{H}_{\mathcal{BH}}$.
- (2) Identifying the near horizon algebra and the Hilbert space of near horizon soft hairs $\mathcal{H}_{\mathcal{NH}}$.
- (3) Providing asymptotic to near horizon embedding map (3.8), which given the commutation relations in the algebras, induces (3.9). This latter in turn provides the relation between the two different presentations of the near horizon algebra in terms of \mathcal{J} or \mathcal{F} 's.

The last step is the crucial part which enables us to construct all states in $\mathcal{H}_{\mathcal{NH}}$ which correspond to the same ‘‘asymptotic black hole state.’’ These states are what we identified as black hole microstates.

There are very rigorous constructions and analysis for the steps (1) and (2). The step (3), while still lacking a rigorous proof, is a very well-justified one. Some early justifications and arguments were already given by Bañados in [45]. Some more comments are

- (1) The asymptotic symmetry algebra, as discussed in [29], is also symplectic and may hence be defined everywhere in the 3d geometry, including at the horizon.
- (2) The near horizon Hilbert space $\mathcal{H}_{\mathcal{NH}}$ which is based on \mathcal{J}_n^\pm contains more states than the asymptotic Hilbert space associated with AdS_3 black holes $\mathcal{H}_{\mathcal{BH}}$. Nonetheless $\mathcal{H}_{\mathcal{NH}}$ and $\mathcal{H}_{\mathcal{BH}}$ are both subsets of the Hilbert space of unitary representations (coadjoint orbits) of Brown-Henneaux Virasoro algebra $\mathcal{H}_{\mathcal{Vir}}$. We note that $\mathcal{H}_{\mathcal{Vir}}$ includes states which correspond to geometries which are not black holes, associated with particles on AdS_3 (conic defects). Specifically,

¹⁵In comparison with the ‘‘fuzzball’’ proposal by S. Mathur [50], we may call this proposal ‘‘fluff ball.’’ See discussions of [1] for the comparison between the two proposals.

- (a) \mathcal{H}_{BH} includes coadjoint orbits whose value of J_0^\pm can be an arbitrary real number. However, as a result of (3.9), we learn that \mathcal{H}_{NH} only contains orbits whose J_0^\pm are “quantized,” as given in (5.4). This latter, as a gain seen from (5.4), means that our horizon fluffs proposal yields a “Bohr-type quantization” on the mass and angular momentum of the corresponding black hole, specifically, $\Delta^\pm \in \mathbb{Z}/c$.
- (b) Moreover, (3.9) and the fact that \mathcal{H}_{NH} is the set of near horizon soft hairs (near horizon zero energy states), implies that \mathcal{H}_{NH} includes a specific subclass of elliptic orbits in \mathcal{H}_{Vir} , those whose ν parameter is quantized as r/c , $r = 1, 2, \dots, c - 1$. Near horizon fluffs, microstates of the given AdS_3 black hole, are states in these orbits.
- (c) If we denote the set of horizon fluffs by \mathcal{H}_F , the above makes it clear that $\mathcal{H}_F \subset \mathcal{H}_{NH} \subset \mathcal{H}_{Vir}$, and that \mathcal{H}_F and \mathcal{H}_{BH} only overlap on the vacuum state.
- (3) The “semi-classical Bohr-type” quantizations are of course confirmed by the specific cases where we know the full quantum theory. That is, for the known cases of the AdS_3/CFT_2 . In the string theory realisations of AdS_3/CFT_2 , e.g., in the D1D5 system [47,48], there are usually some restrictions on the spectrum of states appearing in \mathcal{H}_{Vir} , e.g., the conic defects appear in the spectral flow between R and NS vacua, are expected to have $L_0 = -6r/c$, $r = 0, 1, \dots, c/6$ [48]. (Note that these examples are supersymmetric and the extra factor of 6 compared to our case is due to the fact that we are considering pure AdS_3 gravity case.)

It is of course desirable to make this step (3) as rigorous as the other two steps and make the relation between the \mathcal{H}_{NH} and \mathcal{H}_{Vir} more clear. This will be done in [37].

The analysis of this work, besides extending those of [1] (which was made for BTZ black holes) to generic AdS_3 black holes, clarifies further the proposal and its technical aspects. In particular it makes the construction of the

asymptotic and near horizon Hilbert spaces more explicit. The main result of this work is to identify explicitly the microstates of black holes in the family of Bañados geometries and establish the fact that the entropy is an “orbit invariant” quantity. Of course the orbit invariance of entropy and other thermodynamical properties and the first law of thermodynamics for the Bañados family was already stated and argued for in the semiclassical analysis [28,29] and this work provides a more microscopic description of that.

One of the interesting side results of this work is to show that Bañados geometries are in general coherent states of extended asymptotic symmetry algebra [21]. This is in agreement with the general expectation that geometries, as classical notions, should correspond to coherent states of the underlying quantum system. Here, we of course do not have the full description of pure AdS_3 quantum gravity and its dynamics. However, if this theory makes sense (see [46]), our near horizon Hilbert space \mathcal{H}_{NH} should contain a good part, if not all of, its Hilbert space. In view of the issues with formulation of AdS_3 quantum gravity [46], one may wonder if states which are in \mathcal{H}_{NH} but not in \mathcal{H}_{BH} may be relevant to the resolution of problems faced there.

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