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## Transport coefficients from extremal Gauss-Bonnet black holes

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**ABSTRACT:** We calculate the shear viscosity of strongly coupled field theories dual to Gauss-Bonnet gravity at zero temperature with nonzero chemical potential. We find that the ratio of the shear viscosity over the entropy density is  $1/4\pi$ , which is in accordance with the zero temperature limit of the ratio at nonzero temperatures. We also calculate the DC conductivity for this system at zero temperature and find that the real part of the DC conductivity vanishes up to a delta function, which is similar to the result in Einstein gravity. We show that at zero temperature, we can still have the conclusion that the shear viscosity is fully determined by the effective coupling of transverse gravitons in a kind of theories that the effective action of transverse gravitons can be written into a form of minimally coupled scalars with a deformed effective coupling.

**KEYWORDS:** Gauge-gravity correspondence, Black Holes

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**1 Introduction**

The anti-de Sitter/conformal field theory (AdS/CFT) [1] correspondence has been a quite useful tool in the study of hydrodynamic properties of strongly coupled field theories which have dual gravity descriptions. A remarkable example is the calculation of the ratio of the shear viscosity over the entropy density, which was found to have a universal value  $1/4\pi$  in a variety of theories described by Einstein gravity with or without chemical potential [2–11]. With the leading IIB  $\alpha'^3$  correction at zero chemical potential, the value of the ratio has a positive correction to  $1/4\pi$  [12–18]. This ratio  $1/4\pi$  was conjectured to be a universal lower bound (the KSS bound) [19, 20] for all materials in nature. All known materials in nature by now satisfy this bound. More discussions on the universality and the bound can be found in [21–29].

However, in [30–32] the authors calculated the ratio of the shear viscosity over the entropy density for field theories dual to Gauss-Bonnet gravity using AdS/CFT and found that the ratio has a negative correction proportional to the Gauss-Bonnet coupling constant. After taking into the consideration of causality, a new lower bound  $4/25\pi$  was proposed. Later, in [33] it was conjectured that the value of the shear viscosity is completely determined by the effective coupling of transverse gravitons at the horizon, which was confirmed in [34] and [35] using different methods. The shear viscosity of field theories dual to Gauss-Bonnet gravity coupled to Maxwell fields was also calculated in [36] and in [35] with  $F^4$  corrections, in which the ratio of the shear viscosity over the entropy density obeys the new lower bound. Other progress in this aspect can be found in [37–48].

However, all the calculations above are in the nonzero temperature regime, which are not valid at zero temperature. As argued in a recent paper [49], taking the small frequency (hydrodynamic) limit at zero temperature is subtle as the  $\omega \rightarrow 0$  and  $T \rightarrow 0$  limits do not, in general, commute. Thus one should not just calculate the transport coefficients at nonzero temperatures and after taking the  $T \rightarrow 0$  limit state that this would be the result for the

case of  $T = 0$ . Mathematically, this subtlety arises because the black hole has different singular structures at the horizon at  $T = 0$  from at  $T > 0$ , which lead to different forms of solutions for the hydrodynamic modes. In [49], transport coefficients of field theories with nonzero chemical potentials were calculated at zero temperature in the background of extremal AdS RN black holes using similar methods as in [50] in the calculations of properties of non-Fermi fluid<sup>1</sup> from AdS/CFT, where the IR physics plays an important role in deriving the UV physics on the boundary. They found that at zero temperature, the ratio of the shear viscosity over the entropy density is  $1/4\pi$ , which coincides with the zero temperature limit of the result obtained at nonzero temperatures. It would be more interesting to study the shear viscosity for field theories which are dual to Gauss-Bonnet gravity at zero temperature with nonzero chemical potentials because at nonzero temperatures the ratio of the shear viscosity over the entropy density for duals of Gauss-Bonnet gravity has a nontrivial dependence on the temperature, which is different from the case of Einstein gravity, where the ratio does not depend on the temperature. Also it is interesting to investigate whether the property that the shear viscosity is determined by the effective coupling of transverse gravitons at the horizon still holds at zero temperature. These are the motivations of this paper.

In this paper, we calculate the shear viscosity and DC conductivity for field theories dual to Gauss-Bonnet gravity at zero temperature with nonzero chemical potentials. We find that the ratio of the shear viscosity over the entropy density is  $1/4\pi$ , which does not depend on the Gauss-Bonnet coupling constant and has the same value as that for Einstein gravity, which supports the conjecture made in [49] that the ratio of the shear viscosity over entropy density is a universal value for extremal black holes. We also calculate the DC conductivity for this system up to the first order of the Gauss-Bonnet coupling  $\lambda$  and find that the real part of it vanishes up to a delta function, which is similar to that for Einstein gravity. Furthermore, we use the same methods to show that at zero temperature the shear viscosity is also fully determined by the effective coupling of transverse gravitons in the kind of gravity theories in which the action of transverse gravitons can be written into a form of minimally coupled scalars with a deformed effective coupling which may depend on the radial coordinate. As an application to the specific system of Gauss-Bonnet gravity coupled with Maxwell fields, we show that the effective coupling is the same as Einstein gravity and so the ratio of the shear viscosity over the entropy density is just  $1/4\pi$  at zero temperature, which is consistent with the result we obtain above.

Our paper is organized as follows. In the next section, we calculate the shear viscosity and DC conductivity in the background of an extremal AdS Gauss-Bonnet RN black hole. In section 3, we calculate the shear viscosity with a given effective action of transverse gravitons in the form of a minimally coupled scalar with an effective coupling. section 4 is devoted to conclusions and discussions.

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<sup>1</sup>The original investigation in relating AdS RN black holes and Fermi surfaces by studying thermodynamic and transport behaviors was done in [51], and later also in [52] and [53, 54].

## 2 Transport coefficients from extremal Gauss-Bonnet RN black holes

In this section, we calculate the shear viscosity and conductivity in the background of Extremal Gauss-Bonnet RN black holes in five dimensions. The action of Gauss-Bonnet gravity coupled with Maxwell fields in five dimensions is [55–59]

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R + \frac{12}{\ell^2} + \frac{\lambda}{2} \ell^2 (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2) - \frac{\kappa^2}{2} F_{\mu\nu} F^{\mu\nu} \right), \quad (2.1)$$

where  $\lambda$  is the dimensionless Gauss-Bonnet coupling and  $\kappa^2 = 8\pi G$ . The equations of motion for this action are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{6}{\ell^2} g_{\mu\nu} + \lambda \ell^2 \left[ R R_{\mu\nu} - 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} + R_{\mu\rho\sigma\tau} R_{\nu}{}^{\rho\sigma\tau} - 2R_{\mu\rho} R_{\nu}{}^{\rho} \right. \\ \left. - \frac{1}{4} g_{\mu\nu} (R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2) \right] - \kappa^2 \left( F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F^2 \right) = 0, \quad (2.2)$$

$$\nabla_{\mu} F^{\mu\nu} = 0. \quad (2.3)$$

The AdS Gauss-Bonnet RN black hole solution with a Ricci flat horizon has the form

$$ds^2 = -H(r) N^2 dt^2 + H^{-1}(r) dr^2 + \frac{r^2}{\ell^2} (dx^2 + dy^2 + dz^2), \quad (2.4)$$

$$A = \mu \left( 1 - \frac{r_{\pm}^2}{r^2} \right) dt, \quad (2.5)$$

where

$$H(r) = \frac{r^2}{2\lambda\ell^2} \left[ 1 - \sqrt{1 - 4\lambda \left( 1 - \frac{m}{r^4} + \frac{q^2}{r^6} \right)} \right], \\ N^2 = \frac{1}{2} \left( 1 + \sqrt{1 - 4\lambda} \right), \\ \mu = \frac{\sqrt{6}qN}{2\kappa\ell r_{\pm}^2}.$$

We can rewrite  $H(r)$  as

$$H(r) = \frac{r^2}{2\lambda\ell^2} \left[ 1 - \sqrt{1 - 4\lambda \left( 1 - \frac{r_{\pm}^2}{r^2} \right) \left( 1 - \frac{r_{\mp}^2}{r^2} \right) \left( 1 + \frac{r_{\pm}^2 + r_{\mp}^2}{r^2} \right)} \right],$$

where we have  $m = r_{\pm}^4 + \frac{q^2}{r_{\pm}^2}$ . The thermal properties of this black hole are

$$T = \frac{Nr_{\pm}}{2\pi\ell^2} \left( 2 - \frac{q^2}{r_{\pm}^6} \right), \quad s = \frac{2\pi}{\kappa^2} \left( \frac{r_{\pm}}{\ell} \right)^3, \quad \rho = \sqrt{6} \frac{q}{\kappa\ell^4}, \quad \epsilon = \frac{3N}{2\kappa^2} \frac{m}{\ell^5}. \quad (2.6)$$

It can be easily checked that the first law of thermodynamics  $d\epsilon = Tds + \mu d\rho$  is satisfied.

## 2.1 Extremal Gauss-Bonnet RN black holes

For  $q = \sqrt{2}r_+^3, m = 3r_+^4$ , we have  $T = 0$  and the black hole becomes an extremal one with  $r_+ = r_- = r_0$ . In the following of our calculations it is much more convenient to introduce a dimensionless parameter  $u = r/r_0$ . In the new coordinate system

$$f(u) = \frac{1}{2\lambda} \left[ 1 - \sqrt{1 - 4\lambda \left(1 - \frac{1}{u^2}\right)^2 \left(1 + \frac{2}{u^2}\right)} \right], \quad (2.7)$$

and the solution becomes

$$ds^2 = \frac{u^2 r_0^2}{\ell^2} \left( -f(u) N^2 dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{\ell^2}{u^2} \frac{du^2}{f(u)}, \quad (2.8)$$

$$A = \mu \left( 1 - \frac{1}{u^2} \right) dt, \quad (2.9)$$

where  $\mu = \frac{\sqrt{3}r_0 N}{\kappa \ell}$ . For this extremal black hole (2.8), the horizon is at  $u = 1$  and the boundary is at  $u \rightarrow \infty$ . We introduce  $\alpha = \ell^2/12r_0$  and parameterize the metric by

$$u - 1 = \lambda_1 \frac{\alpha}{\zeta}, \quad t = \lambda_1^{-1} \tau. \quad (2.10)$$

Then we consider the limit  $\lambda_1 \rightarrow 0$  with  $\zeta, \tau$  finite, and we can obtain the following near horizon geometry

$$ds^2 = \frac{\alpha r_0}{\zeta^2} \left[ -N^2 d\tau^2 + d\zeta^2 \right] + \frac{r_0}{12\alpha} \left( dx^2 + dy^2 + dz^2 \right), \quad A = \frac{2\mu\alpha}{\zeta} d\tau, \quad (2.11)$$

clearly it has the structure of  $\text{AdS}_2 \times \text{R}^3$  with a constant electric field.

## 2.2 Shear viscosity

In this and the next subsection, we will calculate the shear viscosity and conductivity of the dual conformal theory to this extremal Gauss-Bonnet black hole. The shear viscosity can be calculated from the Kubo formula using the retarded Green's functions as

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega i} \left( G_{xy,xy}^A(\omega, 0) - G_{xy,xy}^R(\omega, 0) \right), \quad (2.12)$$

where  $\eta$  is the shear viscosity, and the retarded Green's function is defined by

$$G_{\mu\nu,\lambda\rho}^R(k) = -i \int d^4x e^{-ik \cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle. \quad (2.13)$$

The advanced Green's function can be related to the retarded Green's function of energy momentum tensor by  $G_{\mu\nu,\lambda\rho}^A(k) = G_{\mu\nu,\lambda\rho}^R(k)^*$ . In the frame of AdS/CFT correspondence, one is able to compute the retarded Green's function of the stress-energy tensor of the dual field theory by making a small perturbation of the background metric. Here we work in five dimensions and choose spatial coordinates so that the momentum of the perturbation points along the  $z$ -axis. Then the perturbations can be written as  $h_{\mu\nu} = h_{\mu\nu}(t, z, u)$ . In

this basis there are three groups of gravity perturbations in five dimensions, each of which is decoupled from others: the scalar, vector and tensor perturbations [61]. Here we use the simplest one, the tensor perturbation  $h_{xy}$ . We use  $\phi(t, z, u)$  to denote this perturbation with one index raised  $\phi(t, z, u) = h_x^y(t, z, u)$ . To calculate the Green functions of the energy momentum tensor at zero temperature we consider the perturbation  $h_x^y$  in the background of (2.8).

We expand

$$\phi(t, z, u) = \int \frac{dwdp}{(2\pi)^2} e^{-iwt+ipz} \phi(u; k), \quad k = (w, 0, 0, p), \quad \phi(u; -k) = \phi^*(u; k), \quad (2.14)$$

and by plugging this into the action (2.1) we can obtain the quadratic order action for  $\phi(u)$  as

$$S = -\frac{Nr_0^4}{4\kappa^2\ell^5} \int du \frac{dwdp}{(2\pi)^2} u^3 g(u) \left[ \phi'^2(u) - \frac{\ell^4 w^2}{N^2 r_0^2 f^2(u) u^4} \phi^2(u) \right] \quad (2.15)$$

up to some total derivatives,<sup>2</sup> where we have set  $p = 0$  and

$$g(u) = u^2 f(u) \left[ 1 - 2\lambda f(u) - \lambda u f'(u) \right]. \quad (2.16)$$

We can easily derive the equation of motion for  $\phi(u)$  to be

$$\phi''(u) + A_1(u)\phi'(u) + B_1(u)\phi(u) = 0, \quad (2.17)$$

where

$$A_1(u) = \left[ \frac{3}{u} + \frac{g'(u)}{g(u)} \right],$$

$$B_1(u) = \frac{144\alpha^2 w^2}{N^2 f^2(u) u^4}, \quad (2.18)$$

and the prime ' denotes the derivative with respect to  $u$ . The analytic solution of this equation of motion is difficult to obtain, and because we only need to know the boundary behavior of the solution, we follow [49, 50] to divide the background spacetime into two regions and match the solution in the overlapping region. The two regions are defined as

- near region

$$u - 1 = \frac{\alpha w}{\zeta}, \quad \epsilon < \zeta < \infty, \quad (2.19)$$

- far region

$$u - 1 > \frac{\alpha w}{\epsilon}, \quad (2.20)$$

and we consider the limit

$$\alpha w \rightarrow 0, \quad \epsilon \rightarrow 0, \quad \frac{\alpha w}{\epsilon} \rightarrow 0, \quad (2.21)$$

then we can have an overlapping region

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<sup>2</sup>The explicit form of the total derivative terms need not to be written out as has been argued in the appendix of [35] that they would be canceled after taking into the consideration of the Gibbons-Hawking boundary term.

- over-lapping region

$$\zeta \rightarrow 0, \quad u - 1 = \frac{\alpha w}{\zeta} \rightarrow 0. \quad (2.22)$$

In the near region, we can make a coordinate transformation  $u - 1 = \alpha w / \zeta$ , and the coefficients  $A_1(u)$  and  $B_1(u)$  in the equation of motion (2.17) become

$$\begin{aligned} A_1(u) &= \frac{2}{u-1} + \tilde{F}_1(u) = \frac{2\zeta}{\alpha w} + F_1(\zeta), \\ B_1(u) &= \frac{\alpha^2 w^2}{N^2(u-1)^4} + \frac{\alpha^2 w^2 \tilde{G}_1(u)}{(u-1)^3} = \frac{\zeta^4}{N^2 \alpha^2 w^2} + \frac{\zeta^3 G_1(\zeta)}{\alpha w}, \end{aligned} \quad (2.23)$$

where  $\tilde{F}_1(u) = F_1(\zeta)$  and  $\tilde{G}_1(u) = G_1(\zeta)$  are functions regular at  $u = 1$  which do not manifestly depend on  $\alpha w$ . Thus we can expand  $\tilde{F}_1(u)$  and  $\tilde{G}_1(u)$  near  $u = 1$  as

$$\tilde{F}_1(u) = \tilde{F}_1(1) + \tilde{F}'_1(1)(u-1) + \frac{1}{2}\tilde{F}''_1(1)(u-1)^2 + \dots, \quad (2.24)$$

$$\tilde{G}_1(u) = \tilde{G}_1(1) + \tilde{G}'_1(1)(u-1) + \frac{1}{2}\tilde{G}''_1(1)(u-1)^2 + \dots \quad (2.25)$$

and expand

$$F_1(\zeta) = \tilde{F}_1(1) + \tilde{F}'_1(1)\frac{\alpha w}{\zeta} + \frac{1}{2}\tilde{F}''_1(1)\frac{\alpha^2 w^2}{\zeta^2} + \dots, \quad (2.26)$$

$$G_1(\zeta) = \tilde{G}_1(1) + \tilde{G}'_1(1)\frac{\alpha w}{\zeta} + \frac{1}{2}\tilde{G}''_1(1)\frac{\alpha^2 w^2}{\zeta^2} + \dots, \quad (2.27)$$

near  $\alpha w / \zeta = 0$ . Then the equation of motion (2.17) becomes

$$\frac{\partial^2 \phi(\zeta)}{\partial \zeta^2} - \frac{\alpha w}{\zeta^2} F_1(\zeta) \frac{\partial \phi(\zeta)}{\partial \zeta} + \left[ \frac{1}{N^2} + \frac{\alpha w}{\zeta} G_1(\zeta) \right] \phi(\zeta) = 0. \quad (2.28)$$

Because we only need to know the low frequency behavior of the solution in order to get the shear viscosity, we can expand  $\phi(\zeta)$  as

$$\phi(\zeta) = \phi^{(0)}(\zeta) + \alpha w \phi^{(1)}(\zeta) + \alpha^2 w^2 \phi^{(2)}(\zeta) + \dots \quad (2.29)$$

By plugging (2.29) into (2.28) we can obtain the following equation to the leading order

$$\frac{\partial^2 \phi^{(0)}(\zeta)}{\partial \zeta^2} + \frac{1}{N^2} \phi^{(0)}(\zeta) = 0, \quad (2.30)$$

with the following general solutions

$$\phi^{(0)}(\zeta) = a_n^{(0)} e^{i\frac{\zeta}{N}} + b_n^{(0)} e^{-i\frac{\zeta}{N}}. \quad (2.31)$$

The in-falling boundary condition at the horizon gives  $b_n^{(0)} = 0$ . We rewrite the solution (2.31) in the  $u$  coordinate and in the matching region we have

$$\phi^{(0)}(u) = a_n^{(0)} \left\{ [1 + \dots] + \frac{i\alpha w}{(u-1)N} [1 + \dots] \right\}, \quad (2.32)$$

where the dots represent subleading terms. We can also calculate the  $\phi^{(n)}$  terms order by order, but knowing the leading order contribution (2.32) is enough to match the coefficients in the far region.

In the far region, we can also expand  $\phi(u)$  as

$$\phi(u) = \phi^{(0)}(u) + \alpha w \phi^{(1)}(u) + \alpha^2 w^2 \phi^{(2)}(u) + \dots \quad (2.33)$$

By plugging (2.33) into (2.17) we can obtain the following equation to the leading order

$$\frac{\partial^2 \phi^{(0)}(u)}{\partial^2 u} + \left[ \frac{3}{u} + \frac{g'(u)}{g(u)} \right] \frac{\partial \phi^{(0)}(u)}{\partial u} = 0. \quad (2.34)$$

The solution to this equation is

$$\phi^{(0)}(u) = a_f^{(0)} + b_f^{(0)} \left[ -\frac{1}{12(u-1)} + \left( -\frac{1}{18} + 2\lambda \right) \ln(u-1) + Y(u) \right], \quad (2.35)$$

where

$$Y'(u) = \frac{1}{u^3 g(u)} - \frac{1}{12(u-1)^2} - \left( -\frac{1}{18} + 2\lambda \right) \frac{1}{u-1} \quad (2.36)$$

and the coefficient in (2.35) is chosen in order to make sure that  $Y(u)$  is regular at  $u = 1$  and  $\phi^{(0)}(u)$  is regular at  $u \rightarrow \infty$ .

At the boundary  $u \rightarrow \infty$ , the solution (2.35) becomes

$$\phi^{(0)}(u)|_{u \rightarrow \infty} = a_f^{(0)} + b_f^{(0)} \left[ \lim_{u \rightarrow \infty} \left( \left( -\frac{1}{18} + 2\lambda \right) \ln(u-1) + Y(u) \right) - \frac{1 + \sqrt{1-4\lambda}}{8\sqrt{1-4\lambda}} u^{-4} + \dots \right], \quad (2.37)$$

which is the solution we need in order to calculate the shear viscosity. We need to confirm the coefficients of this solution by matching the far region solution (2.35) and the near region solution (2.31) in the matching region.

In the matching region  $u \rightarrow 1$ , the far region solution (2.35) becomes

$$\phi^{(0)}(u)|_{u \rightarrow 1} = \left[ a_f^{(0)} + b_f^{(0)} Y(1) + \dots \right] - \frac{b_f^{(0)}}{12(u-1)} [1 + \dots]. \quad (2.38)$$

By matching (2.32) and (2.38) we have

$$a_f^{(0)} = a_n^{(0)} \left[ 1 + \frac{12i\alpha w}{N} Y(1) \right], \quad b_f^{(0)} = -\frac{12i\alpha w}{N} a_n^{(0)}. \quad (2.39)$$

Substitute (2.39) into (2.37), and we obtain the solution near the boundary as

$$\begin{aligned} \phi(u)|_{u \rightarrow \infty} = & a_n^{(0)} \left[ 1 + \frac{12i\alpha w}{N} \left( Y(1) - \lim_{u \rightarrow \infty} \left( \left( -\frac{1}{18} + 2\lambda \right) \ln(u-1) + Y(u) \right) \right) + \dots \right] \\ & + \frac{3i\alpha w}{2N} a_n^{(0)} \left[ \frac{1 + \sqrt{1-4\lambda}}{\sqrt{1-4\lambda}} + \dots \right] u^{-4}. \end{aligned} \quad (2.40)$$

By plugging the boundary solution (2.40) into the on-shell action for  $\phi(u)$

$$S_{\text{on-shell}} = -\frac{Nr_0^4}{4\kappa^2 \ell^5} \int \frac{dw dp}{(2\pi)^2} u^3 g(u) \left( \phi(u) \phi'(u) \right) \Big|_{u \rightarrow \infty}, \quad (2.41)$$



we obtain

$$\text{Im}G_{xy,xy}^R(w, 0) = -\frac{w}{2\kappa^2} \left(\frac{r_0}{\ell}\right)^3 [1 + \mathcal{O}(w)], \tag{2.42}$$

and using the Kubo-formula we have<sup>3</sup>

$$\eta = -\lim_{w \rightarrow 0} \left( \frac{1}{w} \text{Im}G_{xy,xy}^R(w, 0) \right) = \frac{1}{2\kappa^2} \left(\frac{r_0}{\ell}\right)^3. \tag{2.43}$$

Note that the entropy density  $s = \frac{2\pi}{\kappa^2} \left(\frac{r_0}{\ell}\right)^3$ , so we have

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{2.44}$$

This result is in accordance with the  $T \rightarrow 0$  behavior of the result obtained in [36] for nonzero temperatures though the calculation in [36] is not valid for the  $T = 0$  case. Thus we have the conclusion that for field theories dual to Gauss-Bonnet gravity at zero temperature with chemical potential, the ratio of  $\eta/s$  is also  $1/4\pi$ , which is independent of the Gauss-Bonnet coupling constant and has the same value as that in Einstein gravity. It will be shown at the end of section 3 explicitly that the ratio of  $\eta/s$  we calculate here is the same as in Einstein theory because the ratio of  $\eta/s$  calculated from AdS Gauss-Bonnet black holes depends on the Gauss-Bonnet coupling constant  $\lambda$  only through the product of  $\lambda$  with the  $tt$  component of the background metric  $g_{tt}$  and the first derivative of  $g_{tt}$  on the horizon, and for extremal black holes  $g_{tt}$  and the first derivative of  $g_{tt}$  vanish on the horizon. Thus the ratio of  $\eta/s$  calculated from extremal AdS Gauss-Bonnet black holes does not depend on the parameter  $\lambda$  and is of the same value with the ratio from Einstein gravity, which is different from the non-extremal case.

### 2.3 DC conductivity

In this subsection, we calculate the DC conductivity for the dual field theory in the background of extremal Gauss-Bonnet RN black holes using a similar way as in the last subsection. The DC conductivity  $\sigma$  of the dual field theory can be calculated using Kubo formula from the related Green's function as

$$\sigma = \lim_{\omega \rightarrow 0} \frac{1}{\omega i} G_{x,x}^R(\omega, 0), \tag{2.45}$$

where the retarded Green's function  $G_{x,x}^R(\omega, 0)$  used here is defined as

$$G_{x,x}^R(k) = -i \int d^4x e^{-ik \cdot x} \theta(t) \langle [J_x(x), J_x(0)] \rangle \tag{2.46}$$

and  $J_\mu$  is the current operator dual to the bulk gauge field  $A_\mu$ .

To calculate the retarded Green's function defined above we need to consider small perturbations of the Maxwell field. The perturbation of the electric fields cannot get

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<sup>3</sup>At nonzero temperatures, when we take the  $w \rightarrow 0$  limit we are comparing  $w$  with the temperature  $T$  and in fact taking the  $w/T \rightarrow 0$  limit. At zero temperature, there is another dimensionful parameter: the chemical potential. Thus at  $T = 0$  when we take the limit of  $w \rightarrow 0$ , we are comparing it with the chemical potential and this can be viewed as the  $w/\mu \rightarrow 0$  or  $\alpha w \rightarrow 0$  limit.

decoupled from some metric perturbations. Thus we consider small metric fluctuations for components  $h_t^x(t, z, u)$  and  $h_u^x(t, z, u)$  as well as the electric field fluctuation  $A_x(t, z, u)$ . We expand them as

$$\begin{aligned} h_t^x(t, z, u) &= \int \frac{dwdp}{(2\pi)^2} e^{-iwt+ipz} h_t^x(u; k), \\ h_u^x(t, z, u) &= \int \frac{dwdp}{(2\pi)^2} e^{-iwt+ipz} h_u^x(u; k), \\ A_x(t, z, u) &= \int \frac{dwdp}{(2\pi)^2} e^{-iwt+ipz} a_x(u; k), \end{aligned} \quad (2.47)$$

where we choose  $k = (w, 0, 0, p)$ ,  $\Phi(u; -k) = \Phi^*(u; k)$  with  $\Phi = h_t^x(u; k), h_u^x(u; k), a_x(u; k)$ .

After choosing a gauge  $h_u^x(u) = 0$ , we get the equations of motion for  $a_x(u)$  and  $h_t^x(u)$  as

$$\begin{aligned} \ell^5 \kappa w^2 a_x(u) + Nr_0^2 u f(u) \left[ \ell \kappa N u^3 a_x'(u) f'(u) \right. \\ \left. + 2\sqrt{3} r_0 h_t^{x'}(u) + \ell \kappa N u^2 f(u) (3a_x'(u) + u a_x''(u)) \right] = 0, \end{aligned} \quad (2.48)$$

$$4\sqrt{3} \ell \kappa N a_x(u) + r_0 u^5 (1 - 2\lambda f(u)) h_t^{x'}(u) = 0, \quad (2.49)$$

where the prime ' denotes the derivative with respect to  $u$ . From (2.49) we have

$$h_t^{x'}(u) = -\frac{4\sqrt{3} \ell \kappa N a_x(u)}{r_0 u^5 (1 - 2\lambda f(u))}, \quad (2.50)$$

and by plugging this into (2.48), we obtain the equation of motion for  $a_x(u)$

$$a_x''(u) + A_2(u) a_x'(u) + B_2(u) a_x(u) = 0, \quad (2.51)$$

where

$$A_2(u) = \frac{3}{u} + \frac{f'(u)}{f(u)}, \quad (2.52)$$

$$B_2(u) = \frac{144\alpha^2 w^2}{N^2 u^4 f^2(u)} - \frac{24}{u^8 f(u) (1 - 2\lambda f(u))}. \quad (2.53)$$

We also have to solve this equation by matching the near region solution with the far region solution.

In the near region we make a coordinate transformation  $u - 1 = \alpha w / \zeta$ , then we have

$$\begin{aligned} \tilde{A}_2(\zeta) &= \frac{2}{u-1} + \tilde{F}_2(u) = \frac{2\zeta}{\alpha w} + F_2(\zeta), \\ \tilde{B}_2(\zeta) &= \frac{\alpha^2 w^2}{N^2 (u-1)^4} + \frac{\alpha^2 w^2 \tilde{G}_2(u)}{(u-1)^3} - \frac{2}{(u-1)^2} + \frac{\tilde{H}_2(u)}{(u-1)} \\ &= \frac{\zeta^4}{N^2 \alpha^2 w^2} + \frac{\zeta^3 G_2(\zeta)}{\alpha w} - \frac{2\zeta^2}{\alpha^2 w^2} + \frac{\zeta H_2(\zeta)}{\alpha w}, \end{aligned} \quad (2.54)$$

where  $\tilde{F}_2(u) = F_2(\zeta)$ ,  $\tilde{G}_2(u) = G_2(\zeta)$  and  $\tilde{H}_2(u) = H_2(\zeta)$  are functions regular at  $u = 1$  and do not depend on  $\alpha w$  manifestly. Then we have

$$\tilde{F}_2(u) = \tilde{F}_2(1) + \tilde{F}'_2(1)(u-1) + \frac{1}{2}\tilde{F}''_2(1)(u-1)^2 + \dots, \quad (2.55)$$

$$\tilde{G}_2(u) = \tilde{G}_2(1) + \tilde{G}'_2(1)(u-1) + \frac{1}{2}\tilde{G}''_2(1)(u-1)^2 + \dots, \quad (2.56)$$

$$\tilde{H}_2(u) = \tilde{H}_2(1) + \tilde{H}'_2(1)(u-1) + \frac{1}{2}\tilde{H}''_2(1)(u-1)^2 + \dots, \quad (2.57)$$

and in the  $\zeta$  coordinate

$$F_2(\zeta) = \tilde{F}_2(1) + \tilde{F}'_2(1)\frac{\alpha w}{\zeta} + \frac{1}{2}\tilde{F}''_2(1)\frac{\alpha^2 w^2}{\zeta^2} + \dots, \quad (2.58)$$

$$G_2(\zeta) = \tilde{G}_2(1) + \tilde{G}'_2(1)\frac{\alpha w}{\zeta} + \frac{1}{2}\tilde{G}''_2(1)\frac{\alpha^2 w^2}{\zeta^2} + \dots, \quad (2.59)$$

$$H_2(\zeta) = \tilde{H}_2(1) + \tilde{H}'_2(1)\frac{\alpha w}{\zeta} + \frac{1}{2}\tilde{H}''_2(1)\frac{\alpha^2 w^2}{\zeta^2} + \dots \quad (2.60)$$

In the coordinate  $\zeta$ , the equation of motion (2.51) becomes

$$\frac{\partial^2 a_x(\zeta)}{\partial \zeta^2} - \frac{\alpha w}{\zeta^2} F_2(\zeta) \frac{\partial a_x(\zeta)}{\partial \zeta} + \left[ \frac{1}{N^2} - \frac{2}{\zeta^2} + \frac{\alpha w}{\zeta} \left( G_2(\zeta) + \frac{H_2(\zeta)}{\zeta^2} \right) \right] a_x(\zeta) = 0. \quad (2.61)$$

We now expand  $a_x(\zeta)$  as

$$a_x(\zeta) = a_x^{(0)}(\zeta) + \alpha w a_x^{(1)}(\zeta) + \alpha^2 w^2 a_x^{(2)}(\zeta) + \dots, \quad (2.62)$$

and by plugging (2.62) into (2.61) we obtain the following equation, up to the leading order,

$$\frac{\partial^2 a_x^{(0)}(\zeta)}{\partial \zeta^2} + \left[ \frac{1}{N^2} - \frac{2}{\zeta^2} \right] a_x^{(0)}(\zeta) = 0, \quad (2.63)$$

with the following solution

$$a_x^{(0)}(u) = a_n^{(0)} \left[ 1 + \frac{iN}{\zeta} \right] e^{i\frac{\zeta}{N}} + b_n^{(0)} \left[ 1 - \frac{iN}{\zeta} \right] e^{-i\frac{\zeta}{N}}. \quad (2.64)$$

The in-falling boundary condition at the horizon gives  $b_n^{(0)} = 0$ . In the matching region we can rewrite the near region solution in the  $u$  coordinate as

$$a_x^{(0)}(u) = a_n^{(0)} \left\{ (u-1)[1 + \dots] + \frac{i\alpha^3 w^3}{3(u-1)^2 N^3} [1 + \dots] \right\}. \quad (2.65)$$

The dots denote subleading order contributions and we only need the leading order contributions in the following calculations.

In the far region, we expand  $a_x(u)$  as

$$a_x(u) = a_x^{(0)}(u) + \alpha w a_x^{(1)}(u) + \alpha^2 w^2 a_x^{(2)}(u) + \dots \quad (2.66)$$

By plugging (2.66) into (2.51) we obtain the following equation to the leading order

$$\frac{\partial^2 a_x^{(0)}(u)}{\partial u^2} + A_3(u) \frac{\partial a_x^{(0)}(u)}{\partial u} + B_3(u) a_x^{(0)}(u) = 0, \quad (2.67)$$

where

$$A_3(u) = \frac{3}{u} + \frac{f'(u)}{f(u)},$$

$$B_3(u) = -\frac{24}{u^8 f(u)(1 - 2\lambda f(u))}. \quad (2.68)$$

It is difficult to find the exact analytic solution for the above equation, so we will consider the case of small  $\lambda$  and calculate the first order effect of  $\lambda$  in the following calculations. Now we expand

$$a_x^{(0)}(u) = a_{x0}^{(0)}(u) + \lambda a_{x1}^{(0)}(u) + \mathcal{O}(\lambda^2), \quad (2.69)$$

$$A_3(u) = \frac{3(2 + u^2 + u^4)}{u(-1 + u^2)(2 + u^2)} + \lambda \frac{12(-1 + u^2)}{u^7} + \mathcal{O}(\lambda^2), \quad (2.70)$$

$$B_3(u) = -\frac{24}{u^2(-1 + u^2)^2(2 + u^2)} - \lambda \frac{24}{u^8} + \mathcal{O}(\lambda^2). \quad (2.71)$$

At the order of  $\lambda^0$ , we have

$$\frac{\partial^2 a_{x0}^{(0)}(u)}{\partial u^2} + \frac{3(2 + u^2 + u^4)}{u(-1 + u^2)(2 + u^2)} \frac{\partial a_{x0}^{(0)}(u)}{\partial u} - \frac{24}{u^2(-1 + u^2)^2(2 + u^2)} a_{x0}^{(0)}(u) = 0, \quad (2.72)$$

whose solution is found to be

$$a_{x0}^{(0)} = a_{f0}^{(0)} \left( 1 - \frac{1}{u^2} \right) + b_{f0}^{(0)} \left\{ -\frac{19u^4 - 26u^2 + 10}{54u^2(u^2 - 1)^2} + \frac{4(u^2 - 1)}{81u^2} \ln \frac{u^2 - 1}{u^2 + 2} \right\}. \quad (2.73)$$

At the boundary  $u \rightarrow \infty$ , we have the behavior of the solution as<sup>4</sup>

$$a_{x0}^{(0)}(u)|_{u \rightarrow \infty} = a_{f0}^{(0)} - \left[ a_{f0}^{(0)} + \frac{1}{2} b_{f0}^{(0)} \right] (1 + \dots) u^{-2} \quad (2.74)$$

In the matching region  $u \rightarrow 1$ , we have

$$a_{x0}^{(0)}(u)|_{u \rightarrow 1} = -\frac{b_{f0}^{(0)}}{72(u-1)^2} \left[ 1 + \mathcal{O}(u-1) \right] + (u-1) \left[ 2a_{f0}^{(0)} + \frac{b_{f0}^{(0)}}{648} \left( 231 + 64 \ln \frac{2}{3} \right) + \mathcal{O}(u-1) \right]. \quad (2.75)$$

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<sup>4</sup>Note that there is no logarithmic divergence term in the boundary solution which was shown to exist in [62] because here we are considering the leading contribution and the logarithmic divergence term is a subleading term which comes from the first term in the right hand side of (2.53) and does not affect the result.

On the other hand, at the order of  $\lambda^1$ , the equation of motion for  $a_{x1}^{(0)}(u)$  is

$$\frac{\partial^2 a_{x1}^{(0)}(u)}{\partial u^2} + \frac{3(2+u^2+u^4)}{u(-1+u^2)(2+u^2)} \frac{\partial a_{x1}^{(0)}(u)}{\partial u} - \frac{24}{u^2(-1+u^2)^2(2+u^2)} a_{x1}^{(0)}(u) + \frac{12(-1+u^2)}{u^7} \frac{\partial a_{x0}^{(0)}(u)}{\partial u} - \frac{24}{u^8} a_{x0}^{(0)}(u) = 0, \quad (2.76)$$

whose solution is found to be

$$a_{x1}^{(0)}(u) = \frac{1}{2} b_{f0}^{(0)} + a_{f1}^{(0)} \left(1 - \frac{1}{u^2}\right) + b_{f1}^{(0)} \left\{ -\frac{19u^4 - 26u^2 + 10}{54u^2(u^2 - 1)^2} + \frac{4(u^2 - 1)}{81u^2} \ln \frac{u^2 - 1}{u^2 + 2} \right\} \quad (2.77)$$

In the limit of  $u \rightarrow \infty$ , we have

$$a_{x1}^{(0)}(u)|_{u \rightarrow \infty} = \left(\frac{1}{2} b_{f0}^{(0)} + a_{f1}^{(0)}\right) - \left[a_{f1}^{(0)} + \frac{1}{2} b_{f1}^{(0)}\right] (1 + \dots) u^{-2}. \quad (2.78)$$

While in the matching region  $u \rightarrow 1$ , we have

$$a_{x1}^{(0)}(u)|_{u \rightarrow 1} = -\frac{b_{f1}^{(0)}}{72(u-1)^2} \left[1 + \mathcal{O}(u-1)\right] + (u-1) \left[2a_{f1}^{(0)} + \frac{b_{f1}^{(0)}}{648} \left(231 + 64 \ln \frac{2}{3}\right) + \mathcal{O}(u-1)\right]. \quad (2.79)$$

Matching (2.65) to (2.75) and (2.79) yields

$$\begin{aligned} a_{f0}^{(0)} &= a_n^{(0)} \left[ \frac{1}{2} + \frac{i\alpha^3 w^3}{54} \left(231 + 64 \ln \frac{2}{3}\right) \right], & b_{f0}^{(0)} &= -24i\alpha^3 w^3 a_n^{(0)}, \\ a_{f1}^{(0)} &= a_n^{(0)} \left[ -\frac{i\alpha^3 w^3}{36} \left(231 + 64 \ln \frac{2}{3}\right) \right], & b_{f1}^{(0)} &= -36i\alpha^3 w^3 a_n^{(0)}. \end{aligned} \quad (2.80)$$

Then by substituting (2.80) into (2.74) and (2.78), we have the boundary behavior of the solution as

$$\begin{aligned} a_x(u)|_{u \rightarrow \infty} &= a_n^{(0)} \left[ 1 + i\alpha^3 w^3 \left( \frac{1}{27} \left(231 + 64 \ln \frac{2}{3}\right) - \frac{\lambda}{18} \left(663 + 64 \ln \frac{2}{3}\right) \right) + \dots \right] \\ &+ a_n^{(0)} \left[ -1 + i\alpha^3 w^3 \left( \frac{1}{27} \left(417 - 64 \ln \frac{2}{3}\right) + \frac{\lambda}{18} \left(879 + 64 \ln \frac{2}{3}\right) \right) + \dots \right] u^{-2}. \end{aligned} \quad (2.81)$$

Substitute this solution (2.81) into the on-shell action for  $a_x$ , we obtain

$$S_{\text{on-shell}} = -\frac{Nr_0^2}{2\ell^3} \int \frac{dw dp}{(2\pi)^2} u^3 f(u) (a_x a_x') \Big|_{u \rightarrow \infty}, \quad (2.82)$$

and repeat the procedure in [31], we finally arrive at

$$\text{Im} G_{xx}^R(w, 0) \propto w^3 [1 + \mathcal{O}(w)]. \quad (2.83)$$

As a result, the DC conductivity behaves in the zero frequency limit as

$$\text{Re} \sigma = -\lim_{w \rightarrow 0} \left( \frac{1}{w} \text{Im} G_{xx}^R(w, 0) \right) \propto \omega^2, \quad (2.84)$$

while

$$\text{Im}\sigma = - \lim_{w \rightarrow 0} \left( \frac{1}{w} \text{Re}G_{xx}^R(w, 0) \right) \propto \frac{1}{w}. \quad (2.85)$$

Because of the Kramers-Kronig relation, the real part of the DC conductivity calculated above should have a delta function dependence on  $w$  as a result of a pole in the imaginary part of the conductivity. Thus the real part of the DC conductivity for duals of Gauss-Bonnet gravity at  $T = 0$  should vanish up to a delta function of  $w$ , which is similar to the case in Einstein gravity at  $T = 0$  [49] and is in accordance with [63] in the limit  $T \rightarrow 0$ .

### 3 Universal properties of $\eta/s$ from extremal black holes

In section 2 we have calculated the shear viscosity and the DC conductivity for field theories dual to Gauss-Bonnet gravity at zero temperature with nonzero chemical potential. In this section we consider a kind of field theories which is dual to gravity theories in which the effective action of transverse gravitons can be written into a form of minimally coupled scalars with the coupling constant deformed to an effective coupling whose value generally depends on the radial coordinate. In [33–35], it was shown that for this kind of field theories at nonzero temperatures, the value of the shear viscosity is determined by the value of the effective coupling at the horizon. The explicit form of the effective coupling  $K_{\text{eff}}$  varies for the different gravity theories, which has been calculated for Einstein and Gauss-Bonnet gravity coupled to arbitrary matter fields in a very constrained way in [35].

In this section<sup>5</sup> we will show that for this kind of field theories at zero temperature, we can still have the conclusion that the value of the shear viscosity is proportional to the effective coupling at the horizon. We employ the procedure in [35], but for extremal black holes here. We consider the cases that the perturbation  $h_x^y$  can get decoupled from other modes and first give the assumption of the form of the effective action of transverse gravitons  $h_x^y$  with an effective coupling  $K_{\text{eff}}(v)$  and then calculate the shear viscosity  $\eta$  using Kubo formula for transverse gravitons.

We assume that the effective action of transverse gravitons  $\phi(t, v, z) = h_x^y(t, v, z)$  can be written as

$$S = \frac{1}{2\kappa^2} \int dv \frac{dwdp}{(2\pi)^2} \sqrt{-\bar{g}} \left( K(v) \phi' \phi' + w^2 K(v) \bar{g}_{vv} \bar{g}^{00} \phi^2 - p^2 L(v) \phi^2 \right) \quad (3.1)$$

up to some total derivatives, where a prime stands for the derivative with respect to  $v$ ,  $\bar{g}$  denotes the background metric and

$$\phi(t, v, z) = \int \frac{dwdp}{(2\pi)^2} \phi(v; k) e^{-iwt+ipz}, \quad k = (w, 0, 0, p), \quad \phi(v; -k) = \phi^*(v; k). \quad (3.2)$$

$K(v)$  in this action is related to the effective coupling  $K_{\text{eff}}(v)$  by  $K_{\text{eff}}(v) = K(v) \bar{g}_{vv}$ . It was shown in [35] that for Einstein gravity with arbitrary minimally coupled matter fields,  $K_{\text{eff}}(v) = -1/2$ . Thus  $K_{\text{eff}}(v)$  is a regular function at the horizon  $v = 1$  as long

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<sup>5</sup>In this section we will use the convention of [35], which is different from the one used in the last section. We define  $v = r_0^2/r^2$  and the horizon is located at  $v = 1$  while the boundary is at  $v = 0$ .

as we consider theories with higher derivative corrections which have contributions small compared to the Einstein term. Because  $K_{\text{eff}}(v) = K(v)\bar{g}_{vv}$  and  $K_{\text{eff}}(v)$  is regular at the horizon, we can assume for future use that  $K(v)\sqrt{-\bar{g}} = (1-v)^2M(v)$  for some extremal black hole backgrounds, where  $M(v)$  is a function regular at the horizon.

Because the shear viscosity only involves physics in the zero momentum limit,  $L(v)$  would not affect the value of  $\eta$ . The only constraint on  $L(v)$  is that it should be regular at the horizon  $v = 1$ . In fact we can also have an extra term  $w^2N(v)\phi^2$  in the action (3.1), and we assume  $N(v)$  is also a function of  $v$ , regular at the horizon  $v = 1$ . The addition of such a term will not affect the value of  $\eta$ .

Because we are working at field theories at zero temperature, the background metric  $\bar{g}$  should be an extremal black hole which is assumed to be

$$ds^2 = -y(v)(1-v)^2 dt^2 + \frac{1}{h(v)(1-v)^2} dv^2 + \frac{r_0^2}{v\ell^2} (d\vec{x}^2), \quad (3.3)$$

where  $y(v)$  and  $h(v)$  take finite values at  $v = 1$ . In this coordinate system the boundary is at  $v = 0$  and the horizon is at  $v = 1$ . This coordinate system can be related to the ordinary one by a transformation  $v = r_0^2/r^2$ . The equation of motion for the transverse gravitons derived from (3.1) is

$$\phi''(v, k) + A(v)\phi'(v, k) + B(v)\phi(v, k) = 0, \quad (3.4)$$

where

$$A(v) = \frac{(K(v)\sqrt{-\bar{g}})'}{K(v)\sqrt{-\bar{g}}}, \quad (3.5)$$

and

$$B(v) = -\bar{g}_{vv}\bar{g}^{00}w^2 + \frac{L(v)}{K(v)}p^2. \quad (3.6)$$

In the  $p \rightarrow 0$  limit

$$B(v) = \frac{w^2}{y(v)h(v)(1-v)^4}. \quad (3.7)$$

The behaviors of  $A(v)$  and  $B(v)$  near the horizon  $v = 1$  are quite different from those in the case of non-extremal black holes. Thus in the near region, the behavior of  $\phi(v)$  is different from that in [35]. We define the near region to be  $1-v \ll 1$  and the far region to be  $1-v \gg \sqrt{\alpha w}$ , where  $\alpha$  is the same as defined in last section. In the  $\alpha w \ll 1$  limit, there exists a matching region  $\sqrt{\alpha w} \ll 1-v \ll 1$ .

In the near region  $1-v \ll 1$ , the equation of motion (3.4) becomes

$$\phi''(v) - \frac{2}{1-v}\phi'(v) + \frac{w^2}{y(1)h(1)(1-v)^4}\phi(v) = 0. \quad (3.8)$$

The solution to this equation has the form

$$\phi(v) = ae^{\frac{iw}{\sqrt{y(1)h(1)(1-v)}}} + be^{-\frac{iw}{\sqrt{y(1)h(1)(1-v)}}}, \quad (3.9)$$

where  $a$  and  $b$  are two arbitrary constants. The in-falling boundary condition sets  $b = 0$ . On the other hand, in the far region  $1 - v \gg \sqrt{\alpha w}$ , the equation of motion (3.4) becomes

$$\phi''(v) + A(v)\phi'(v) = 0. \tag{3.10}$$

This equation has the solution as

$$\phi(v) = \int \frac{C}{K(v)\sqrt{-g}} dv + C_0, \tag{3.11}$$

where  $C$  and  $C_0$  are two integration constants.

Note that we have defined  $K(v)\sqrt{-g} = (1 - v)^2 M(v)$ , where  $M(v)$  a function regular at  $v = 1$ . According to the factorization rule of rational fractions,  $C/(K(v)\sqrt{-g})$  can be written as

$$\frac{C}{(1 - v)^2 M(v)} = \frac{C_1}{(1 - v)^2} - \frac{C_2}{1 - v} + X(v), \tag{3.12}$$

where  $X(v)$  is a function regular at  $v = 1$ . From the regularity of  $X(v)$  at  $v = 1$  we can fix the constants  $C_1$  and  $C_2$  to be

$$C_1 = \frac{C}{M(1)}, \quad C_2 = \frac{CM'(1)}{M^2(1)}. \tag{3.13}$$

Thus  $\phi(v)$  can be integrated out to be

$$\phi(v) = \frac{C}{M(1)(1 - v)} + \frac{CM'(1)}{M^2(1)} \ln(1 - v) + Z(v), \tag{3.14}$$

where  $Z(v)$  is a function regular at  $v = 1$  and

$$Z'(v) = \frac{C}{(1 - v)^2} \left( \frac{1}{M(v)} - \frac{1}{M(1)} \right) + \frac{CM'(1)}{M^2(1)(1 - v)}. \tag{3.15}$$

Then in the matching region  $\sqrt{\alpha w} \ll 1 - v \ll 1$ , the near region solution (3.9) can be expanded to the first order of  $\alpha w/(1 - v)$  as

$$1 + \frac{iw}{\sqrt{y(1)h(1)}(1 - v)}, \tag{3.16}$$

where we have chosen the normalization constant  $a$  to be one, while the far region solution in the matching region is still of the form of (3.14). Thus we have

$$\frac{C}{M(1)} = \frac{iw}{\sqrt{y(1)h(1)}}, \tag{3.17}$$

and  $Z(v) = 1 + iwz(v)$  by matching the leading terms of the near and far region solutions. Note that the sub-leading terms in the far region solution (3.14) need not to appear in the near region solution as it can be easily checked that the near region solution cannot be trusted at this order. Thus the solution near the boundary is

$$\phi(v) = 1 + \frac{iw}{\sqrt{y(1)h(1)}(1 - v)} + \frac{iwM'(1)}{M(1)\sqrt{y(1)h(1)}} \ln(1 - v) + iwz(v), \tag{3.18}$$



where

$$z'(v) = \frac{M(1)}{\sqrt{y(1)h(1)}(1-v)^2} \left( \frac{1}{M(v)} - \frac{1}{M(1)} \right) + \frac{M'(1)}{M(1)\sqrt{y(1)h(1)}(1-v)^2}. \quad (3.19)$$

Substituting the solution (3.18) into the on-shell action

$$S_{\text{on-shell}} = \frac{1}{2\kappa^2} \int \frac{dw dp}{(2\pi)^2} dv \left( \sqrt{-\bar{g}} K(v) \phi' \phi \right)', \quad (3.20)$$

and integrating this action gives

$$S_{\text{on-shell}} = \frac{1}{2\kappa^2} \int \frac{dw dp}{(2\pi)^2} \left( \sqrt{-\bar{g}} K(v) \phi' \phi \right) \Big|_{v=0}^{v=1}. \quad (3.21)$$

As argued in the appendix of [35], the total derivative terms and the Gibbons-Hawking contribution exactly cancel and (3.21) is the total contribution. With the help of the boundary term, we have

$$G_{xy,xy}^R(w, 0) = -\frac{1}{2\kappa^2} 2\sqrt{-\bar{g}} K(v) \phi'^* \phi|_{v=0}. \quad (3.22)$$

Substituting the solution (3.18) into the Kubo formula we finally reach

$$\eta = \frac{1}{2\kappa^2} \lim_{w \rightarrow 0} \frac{2\sqrt{-\bar{g}} K(v) \phi'^* \phi|_{v=0}}{iw} = \frac{1}{2\kappa^2} \left( \frac{r_0}{\ell} \right)^3 \left( -2K_{\text{eff}}(v=1) \right). \quad (3.23)$$

Thus we get to the conclusion that the shear viscosity is fully determined by the value of the effective coupling  $K_{\text{eff}}(v)$  at the horizon. This indicates that the shear viscosity is totally determined by the IR physics, as we are considering the  $w \rightarrow 0$  limit which encodes only the near horizon physics.

The form of  $K_{\text{eff}}(v)$  for some very limited class of gravity theories have been obtained in [35]. For Einstein gravity coupled with matter fields in a constrained way [35],  $K_{\text{eff}} = -1/2$ , so the ratio of  $\eta/s$  is always  $1/4\pi$  because in Einstein gravity the entropy density is proportional to the horizon area. For Gauss-Bonnet gravity coupled with matter fields in the same constrained way,  $K_{\text{eff}}(v)$  was calculated for a specific kind of background metric

$$ds^2 = -\frac{g^*(v)r_0^2 N^2}{\ell^2 v} dt^2 + \frac{\ell^2}{4v^2 g^*(v)} dv^2 + \frac{r_0^2}{v\ell^2} d\vec{x}^2, \quad (3.24)$$

it is

$$K_{\text{eff}}(v) = -\frac{1}{2} \left[ 1 - 2\lambda g^*(v) + 2\lambda v g^{*'}(v) \right]. \quad (3.25)$$

This form of metric is in fact the kind of metric which has  $g_{tt}g_{rr} = -1$  in the ordinary  $r$  coordinate, which is satisfied by most of the metrics we are interested in. From the formula (3.25) we can see that the value of the shear viscosity has a dependence on  $\lambda$  only through the product of  $\lambda$  with  $g^*(1)$  and  $g^{*'}(1)$  which vanish for extremal black holes. Thus in this specific kind of background metric we find that  $K_{\text{eff}}(v) = -1/2$  for extremal black holes, which means that  $\eta/s = 1/4\pi$  for the cases that the entropy density is proportional to the horizon area [55–57]. This result is the same as in Einstein gravity as it has no dependence on the Gauss-Bonnet coupling  $\lambda$  as we explained above and is in accordance with the result obtained in last section.

## 4 Conclusion and discussion

In this paper we calculated the ratio of the shear viscosity over the entropy density for strongly coupled field theories dual to Gauss-Bonnet gravity at zero temperature with a nonzero chemical potential. We found that the ratio is  $1/4\pi$ , which does not depend on the Gauss-Bonnet coupling constant and has the same value as that for Einstein gravity. We also calculated the DC conductivity for this system up to the first order of the Gauss-Bonnet coupling  $\lambda$  and found that the real part of it vanishes up to a delta function, which is similar to the case of Einstein gravity, again.

We showed that the value of the shear viscosity depends on the effective coupling of transverse gravitons for a kind of gravity theories in which the effective action of transverse gravitons can be written into a form of minimally coupled scalars with a deformed effective coupling. The effective coupling calculated in Einstein and Gauss-Bonnet gravity theories with minimally coupled matter fields shows that the ratio of  $\eta/s$  in such theories is always  $1/4\pi$  at zero temperature. The results obtained in this paper combining with those from [49] indicate that those values of shear viscosity and conductivity are universal for conformal field theories with gravity duals which have minimally coupled matter fields at zero temperature.<sup>6</sup> This is also closely related to the fact that for extremal black holes there exists an  $\text{AdS}_2$  factor in the near horizon geometry. In this sense, it would be of great interest to calculate other transport coefficients for extremal black holes.

The shear viscosity is fully determined by the near horizon value of the effective coupling of transverse gravitons. However, we still have very poor knowledge of the exact form of  $K_{\text{eff}}$  in various gravity theories. It would be very interesting to calculate this quantity in more general gravity theories and in more general kinds of backgrounds, such as the black hole background in the non-relativistic version of AdS/CFT.

**Note added.** After this work appeared on arXiv, we notice that a paper [64] appeared on the same day, which discussed the same topic and reached similar conclusions with ours. On the next day another paper [65] appeared which has some overlap with our present paper.

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<sup>6</sup>In the case of gravity with nonminimally coupled matter fields, at  $T = 0$  there may be a violation of the universality of  $\eta/s$ , which can be seen in [44] at the limit  $T \rightarrow 0$ , though the  $T \rightarrow 0$  limit may not give the result at  $T = 0$ .

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