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Generalized massive gravity in AdS_3 spacetime

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In this note we investigate the generalized massive gravity in asymptotically AdS_3 spacetime by combining the two mass terms of topological massive gravity and new massive gravity theory. We study the linearized excitations around the AdS_3 background and find that at a specific value of a certain combination of the two mass parameters (chiral line), one of the massive graviton solutions becomes the left-moving massless mode. It is shown that the theory is chiral at this line under Brown-Henneaux boundary condition. Because of this degeneration of the gravitons the new log solution which has a logarithmic asymptotic behavior is also a solution to this gravity theory at the chiral line. The log boundary condition which was proposed to accommodate this log solution is proved to be consistent at this chiral line. The resulting theory is no longer chiral except at a special point on the chiral line, where another new solution with log-square asymptotic behavior exists. At this special point, we prove that a new kind of boundary condition called log-square boundary condition, which accommodates this new solution, can be consistent.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–4] gives much insight into the quantum theory of gravity and the central idea is that what it means to solve the quantum gravity with a negative cosmological constant is to find the dual conformal field theory. It would be interesting to test this idea in three dimensional pure quantum gravity with a negative cosmological constant and find as much useful information of the dual conformal field theory from the gravity side. The central charge of the corresponding conformal field theory has been derived in [5] two decades ago from the observation that the asymptotic symmetry group of AdS_3 under Brown-Henneaux boundary conditions possesses two sets of Virasoro algebras and the Hilbert space should be a representation of this algebra. Much work has been done on AdS_3/CFT_2 to gain valuable insights into the quantization of gravity in asymptotic AdS_3, which can be found in [6,7] and references therein.

Over the past two years, two remarkable progresses have been made in AdS_3/CFT_2. One is for pure gravity in AdS_3 and the CFT dual of this quantum gravity has been identified in [8,9]. The other progress is that the field theory dual for a Chern-Simons (CS) deformation of pure Einstein gravity theory named topological massive gravity (TMG) [10,11] with a negative cosmological constant has been investigated and it is found that there exists a chiral point at which the theory becomes chiral with only right-moving modes [12]. TMG has been further discussed in [13–36].

In [37] another interesting massive deformation of pure gravity has been proposed in three dimensions. In this new massive gravity (NMG), higher derivative terms are added to the Einstein-Hilbert action and unlike in topological massive gravity, parity is preserved in this new massive gravity. This new massive gravity is equivalent to the Pauli-Fierz action for a massive spin-2 field at the linearized level in asymptotically Minkowski spacetime. In [38,39], the unitarity of this new massive gravity and the new massive gravity with a Pauli-Fierz mass term was examined. Black hole solutions for this new massive gravity with a negative cosmological constant have been considered in [40]. In [41], the linearized gravitational excitations of this new massive gravity around asymptotically AdS_3 spacetime has been studied and it was also found that there is a critical point for the mass parameter. It was conjectured in [41] that this theory is trivial at the critical point under Brown-Henneaux boundary condition, and was later proved in [42]. Other consistent boundary conditions are also studied in [42].

TMG and NMG have much in common, and indeed can be unified into a general massive gravity theory [37] (GMG). The generalized massive gravity theory is realized by adding both the CS deformation term and the higher derivative deformation term to pure Einstein gravity with a negative cosmological constant. This theory has two mass parameters and TMG and NMG are just two different limits of this generalized theory. This theory is expected to have more interesting physics because we can have one more adjustable mass parameter.

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In this note we will study this generalized massive gravity theory in AdS_3. We begin with investigations on the behavior of linearized excitations of GMG in asymptotically AdS_3. We show that there exists a critical line of the two mass parameters at which one of the massive graviton becomes massless like what happens at the chiral point in TMG. At this critical line the new solution with log asymptotic behavior which was found in [15] for TMG is also a solution to this GMG. The linearized behavior in this theory is different from that of TMG in that there also exists a special point on the critical line at which both of the two massive gravitons become left-moving massless gravitons and new interesting physics arises. This special point is also different from in NMG, where parity is preserved and at the special point the two massive gravitons become one left-moving massless and one right-moving massless modes. At this special point in GMG, another new solution which has log-square asymptotic behavior appears.

The GMG theory at the critical line has zero left-moving central charge and is shown to be chiral under Brown-Henneaux boundary condition at the linearized level. Because of the appearance of the new log solution, we try to relax the boundary condition to the log boundary condition and find that this boundary condition is consistent with GMG at the chiral line. The theory is no longer chiral under the log boundary condition except at the special point mentioned above. At the special point, with the hint from the asymptotic behavior of the new solution which does not obey either the Brown-Henneaux boundary condition or the log boundary condition, we can further relax the log boundary condition to the new log-square boundary condition and find it is indeed consistent at this point.

Our note is organized as follows. In Sec. II we will formulate the generalized massive gravity in asymptotically AdS_3 and consider the linearized excitations. In Sec. III we will first write out the formula for the calculation of conserved charges in this theory. Then we will examine the consistency of the Brown-Henneaux boundary condition, the Log boundary condition at the chiral line and the log-square boundary condition at the special point. Sec. IV is devoted to conclusions and discussions.

II. THE GENERALIZED MASSIVE GRAVITY THEORY

In this section we will first write out the action of the generalized massive gravity theory which is obtained by combining the two mass terms of TMG and NMG. We will show that the theory can be treated as a generalized massive gravity in asymptotically AdS_3 in the sense that there are two independent mass parameters in the linearized equation of motion after gauge fixing. This is similar to the case without a cosmological constant discussed in [37]. Then we will study the linearized solutions of this generalized massive gravity around AdS_3.

A. The generalized massive gravity theory

The action for the generalized massive gravity theory [37] can be written as[1]

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[(R-2\lambda) + \frac{1}{\mu}\mathcal{L}_{CS} - \frac{1}{m^2}K\right], \tag{2.1}$$

where

$$K = R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2, \tag{2.2}$$

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\lambda\mu\nu}\Gamma^{\alpha}_{\lambda\sigma}\left[\partial_\mu\Gamma^{\sigma}_{\alpha\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\alpha}\right], \tag{2.3}$$

m, μ are the two mass parameters of this generalized massive gravity and λ is a constant which is different from the cosmological constant of the AdS_3 background solution. The sign of m^2 is not fixed here and it can be either positive or negative. We assume μ to be positive and cases with negative μ can be obtained by exchanging the coordinates τ^+ and τ^-, whose definition can be found below.

The equation of motion of this action is

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{2.4}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{2.5}$$

$$\begin{aligned} K_{\mu\nu} = &-\frac{1}{2}\nabla^2 R g_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R + 2\nabla^2 R_{\mu\nu} \\ &+ 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} \\ &+ \frac{3}{8}R^2 g_{\mu\nu}, \end{aligned} \tag{2.6}$$

and the Cotton tensor

$$C_{\mu\nu} = \varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha\left(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R\right). \tag{2.7}$$

One special feature of this choice of K is that $g^{\mu\nu}K_{\mu\nu} = K$.

After introducing a nonzero λ, the new massive gravity theory could have an AdS_3 solution

[1] We take the metric signature $(-,+,+)$ and follow the notation and conventions of MTW [43]. G is the three dimensional Newton constant which is positive here. We take $\varepsilon_{\mu\nu\alpha} = \sqrt{-g}\epsilon_{\mu\nu\alpha}$ with $\epsilon_{012} = -1$. Our convention is the same as [12].

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \ell^2(-\cosh^2\rho d\tau^2 + \sinh^2\rho d\phi^2 + d\rho^2), \tag{2.8}$$

while the λ in the action should be related to the cosmological constant Λ and the mass parameter by

$$m^2 = \frac{\Lambda^2}{4(-\lambda + \Lambda)}, \tag{2.9}$$

and

$$\Lambda = -1/\ell^2. \tag{2.10}$$

Note that for a given λ, there may exist two AdS_3 solutions [37] with different AdS radii. We can expand around either one of them and denote ℓ as the AdS radius of the one we expand around. The other solution has different asymptotic behavior and will not affect our result.

It would be useful to introduce the light-cone coordinates $\tau^\pm = \tau \pm \phi$, then the AdS_3 spacetime (2.8) could be written as

$$ds^2 = \frac{\ell^2}{4}(-d\tau^{+2} - 2\cosh 2\rho d\tau^+ d\tau^- - d\tau^{-2} + 4d\rho^2). \tag{2.11}$$

The central charges [7,44–46] for this theory in asymptotically AdS_3 are

$$c_L = \frac{3\ell}{2G}\left(1 - \frac{1}{2m^2\ell^2} - \frac{1}{\mu\ell}\right), \quad c_R = \frac{3\ell}{2G}\left(1 - \frac{1}{2m^2\ell^2} + \frac{1}{\mu\ell}\right). \tag{2.12}$$

To have nonnegative central charges we have to impose the constraints $c_L \geq 0$ and $c_R \geq 0$ on the mass parameters.

B. Linearized equation of motion

By expanding $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ around the AdS_3 background solution (2.8), we could obtain the equation of motion for the linearized excitations $h_{\mu\nu}$ as

$$G^{(1)}_{\mu\nu} + \lambda h_{\mu\nu} + \frac{1}{\mu}C^{(1)}_{\mu\nu} - \frac{1}{2m^2}K^{(1)}_{\mu\nu} = 0, \tag{2.13}$$

where

$$R^{(1)}_{\mu\nu} = \frac{1}{2}(-\bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\sigma\nu}), \tag{2.14}$$

$$R^{(1)} \equiv (R_{\mu\nu}g^{\mu\nu})^{(1)} = -\bar{\nabla}^2 h + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - 2\Lambda h, \tag{2.15}$$

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} - 3\Lambda h_{\mu\nu}, \tag{2.16}$$

$$K^{(1)}_{\mu\nu} = -\frac{1}{2}\bar{\nabla}^2 R^{(1)}\bar{g}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu R^{(1)} + 2\bar{\nabla}^2 R^{(1)}_{\mu\nu} - 4\Lambda\bar{\nabla}^2 h_{\mu\nu} - 5\Lambda R^{(1)}_{\mu\nu} + \frac{3}{2}\Lambda R^{(1)}\bar{g}_{\mu\nu} + \frac{19}{2}\Lambda^2 h_{\mu\nu}, \tag{2.17}$$

$$C^{(1)}_{\mu\nu} = \varepsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\left(R^{(1)}_{\beta\nu} - \frac{1}{4}\bar{g}_{\beta\nu}R^{(1)} - 2\Lambda h_{\beta\nu}\right). \tag{2.18}$$

We can also have $R^{(1)} = 0$ by tracing both sides of the equation of motion of gravitons (2.13). Thus we can choose the gauge to be $h = 0$ and $\nabla_\mu h^{\mu\nu} = 0$ simultaneously.[2] After gauge fixing, the equation of motion becomes

$$(\bar{\nabla}^2 - 2\Lambda)\left(\bar{\nabla}^2 h_{\mu\nu} - \frac{m^2}{\mu}\varepsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha h_{\beta\nu} - \left(m^2 + \frac{5}{2}\Lambda\right)h_{\mu\nu}\right) = 0. \tag{2.19}$$

We can define the following four operators [12,37] which commute with each other as

$$(\mathcal{D}^{L/R})_\mu{}^\nu = \delta^\nu_\mu \pm \ell\varepsilon_\mu{}^{\alpha\nu}\bar{\nabla}_\alpha, \tag{2.20}$$

$$(\mathcal{D}^{m_i})_\mu{}^\nu = \delta^\nu_\mu + \frac{1}{m_i}\varepsilon_\mu{}^{\alpha\nu}\bar{\nabla}_\alpha, \quad i = 1, 2, \tag{2.21}$$

and the linearized equation of motion can be written using these four operators to be

$$(\mathcal{D}^L\mathcal{D}^R\mathcal{D}^{m_1}\mathcal{D}^{m_2}h)_{\mu\nu} = 0, \tag{2.22}$$

with $m_1 m_2 = -m^2 - \frac{1}{2\ell^2}$ and $m_1 + m_2 = -\frac{m^2}{\mu}$. At this linearized level, the generalized massive gravity has two mass parameters m_1 and m_2 which can be different from each other. The value of $1/\mu$ reflects the extent to which parity is violated in this generalized massive gravity. When $\mu \to \infty$, $m_1 = -m_2$ and the linearized equation of motion (2.22) goes back to the case of NMG with parity preserved. When $m^2 \to \pm\infty$, one of m_i goes to infinity and the other is equal to μ, so one of the operator $\mathcal{D}^{m_i}$ becomes the identity operator and the linearized equation of motion (2.22) goes back to the case of TMG. Thus we can see that TMG is a special case of this generalized massive gravity in that one of the mass parameter goes to infinity and NMG is a special case of this generalized massive

[2] In fact, we cannot have $R^{(1)} = 0$ at the point $m^2\ell^2 = -1/2$, but we do not need to worry about this because it will be shown in the next subsection that this point is excluded from the parameter region that we are interested in.

gravity in that the two mass parameters are equal in value while with different signs. This gravity theory can be viewed as a generalized massive gravity in the sense that the two mass parameters m_1 and m_2 in the linearized equation of motion can be independent with each other.

Therefore, because this generalized massive gravity is a more generalized theory than TMG and NMG, it would be interesting to investigate the consistent boundary conditions for this theory and in the remainder of this paper we will focus on this generalized massive gravity and found that there are other special points where novel properties arise that the ones in TMG and NMG because m_1 and m_2 here can be chosen arbitrarily.

C. Linearized solutions

We solve for the highest weight states which obey $L_0|\psi_{\mu\nu}\rangle = h|\psi_{\mu\nu}\rangle$ and $\bar{L}_0|\psi_{\mu\nu}\rangle = \bar{h}|\psi_{\mu\nu}\rangle$. We can have the following four sets of solutions

$$
\begin{aligned}
(h,\bar{h}) &= (2,0), \qquad (h,\bar{h}) = (0,2),\\
(h,\bar{h}) &= \left(\frac{6-\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}, \frac{-2-\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}\right),\\
(h,\bar{h}) &= \left(\frac{-2+\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}, \frac{6+\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}\right),
\end{aligned}
$$

where

$$
\psi_{\mu\nu}(h,\bar{h}) = f(\rho,\tau^+,\tau^-)\begin{pmatrix} 1 & \frac{h-\bar{h}}{2} & \frac{i}{\sinh(\rho)\cosh(\rho)} \\ \frac{h-\bar{h}}{2} & 1 & \frac{i(h-\bar{h})}{2\sinh(\rho)\cosh(\rho)} \\ \frac{i}{\sinh(\rho)\cosh(\rho)} & \frac{i(h-\bar{h})}{2\sinh(\rho)\cosh(\rho)} & -\frac{1}{\sinh^2(\rho)\cosh^2(\rho)} \end{pmatrix}, \tag{2.23}
$$

and $f(\rho,\tau^+,\tau^-) = e^{-ih\tau^+ - i\bar{h}\tau^-}(\cosh\rho)^{(-h-\bar{h})}\sinh^2\rho$.

The first two solutions are the left and right-moving massless gravitons, respectively, which are solutions of $(\mathcal{D}^L h)_{\mu\nu} = 0$ and $(\mathcal{D}^R h)_{\mu\nu} = 0$. The third and fourth solutions are massive gravitons which are solutions of $(\mathcal{D}^{m_1} h)_{\mu\nu} = 0$ and $(\mathcal{D}^{m_2} h)_{\mu\nu} = 0$. To avoid any exponential divergence at the boundary, the third and fourth solutions reduce to the following solutions with the following parameter regions:

(i) $m^2 > 0$ and $c_L \geq 0$

$$
\begin{aligned}
(h,\bar{h}) &= \left(\frac{6-\frac{m^2\ell}{\mu} + \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}, \frac{-2-\frac{m^2\ell}{\mu} + \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}\right),\\
(h,\bar{h}) &= \left(\frac{-2+\frac{m^2\ell}{\mu} + \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}, \frac{6+\frac{m^2\ell}{\mu} + \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}\right);
\end{aligned}
$$

(ii) $\mu\ell \geq \frac{3}{4}$ and $m^2\ell \leq -2\mu^2\ell - \mu\sqrt{4\mu^2\ell^2 - 2}$ and $c_L \geq 0$

$$
(h,\bar{h}) = \left(\frac{6-\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}, \frac{-2-\frac{m^2\ell}{\mu} \pm \sqrt{2+4m^2\ell^2+\frac{m^4\ell^2}{\mu^2}}}{4}\right);
$$

(iii) Otherwise: we only have massless gravitons which do not blow up at the boundary.

For the mass of the graviton to be nonnegative, we also have to impose the same conditions $\mu\ell \geq \frac{3}{4}$ and $m^2\ell \leq -2\mu^2\ell - \mu\sqrt{4\mu^2\ell^2 - 2}$ or $m^2 \geq 0$ along with the conditions $c_L \geq 0$. Thus in the remainder of this paper, we only concentrate within this parameter region.

At the critical line $c_L = 0$, the linearized equation of motion defined using the mutually commuting operators (2.22) becomes

$$(\mathcal{D}^R\mathcal{D}^L\mathcal{D}^{m_1}\mathcal{D}^L h)_{\mu\nu}=0, \tag{2.24}$$

where $m_1=-\frac{1+2m^2\ell^2}{2\ell}$. From this expression we can see that two of the operators degenerate here and the solutions degenerate here, too. Thus we can construct the new "log" solution at $c_L=0$ following [15]. For $m^2>0$, the left-moving massless and massive graviton solutions can be used to construct a new log solution while for $\mu\ell\geq\frac{3}{4}$ and $m^2\ell\leq -2\mu^2\ell-\mu\sqrt{4\mu^2\ell^2-2}$, the new log solution can be obtained using the solution with the minus sign in front of the square root. The log solutions constructed in these two ways are the same and we have

$$\psi^{\mathrm{new}L}_{\mu\nu}=y(\tau,\rho)\psi^L_{\mu\nu}, \tag{2.25}$$

where $y(\tau,\rho)=-i\tau-\ln\cosh\rho$ and $\psi^L_{\mu\nu}$ is the wave function for left-moving massless mode. The new solution satisfies

$$(\mathcal{D}^L\mathcal{D}^L h^{\mathrm{new}L})_{\mu\nu}=0,\ (\mathcal{D}^L h^{\mathrm{new}L})_{\mu\nu}\neq 0. \tag{2.26}$$

Note here that though this new solution was constructed within the parameter regions shown above, the solution is a solution to (2.24) for arbitrary value of the mass parameters obeying $c_L=0$.

At $c_L=0$, there is a special point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$, at which $\mathcal{D}^{m_1}$ also becomes $\mathcal{D}^L$. The linearized equation of motion becomes

$$(\mathcal{D}^R\mathcal{D}^L\mathcal{D}^L\mathcal{D}^L h)_{\mu\nu}=0. \tag{2.27}$$

Thus the two solutions of massive gravitons both coincide with the left-moving massless solution. Remember that in the case of NMG theory [41], at the point $m^2\ell^2=1/2$ the two modes of massive gravitons also become massless gravitons simultaneously. However, in that theory, the two massive gravitons become one left-moving and one right-moving massless modes. Here, because of a nonzero μ, parity is violated and the two massive modes can be both reduced to two left-moving massless modes at a certain point.

Now because of this degeneration new solutions could appear which obey

$$\begin{aligned}(\mathcal{D}^L\mathcal{D}^L\mathcal{D}^L h^{\mathrm{new}S})_{\mu\nu}&=0,\\ (\mathcal{D}^L\mathcal{D}^L h^{\mathrm{new}S})_{\mu\nu}&\neq 0,\\ (\mathcal{D}^L h^{\mathrm{new}S})_{\mu\nu}&\neq 0,\end{aligned} \tag{2.28}$$

and we find one such solution to be

$$\psi^{\mathrm{new}S}_{\mu\nu}=Y(\tau,\rho)\psi^L_{\mu\nu}, \tag{2.29}$$

where $Y(\tau,\rho)=(-i\tau-\ln\cosh\rho)^2$ and $\psi^L_{\mu\nu}$ is the wave function of the left-moving massless mode. This new solution does not obey either Brown-Henneaux boundary conditions or log boundary conditions. In order to accommodate this new solution we will have to impose an even looser boundary condition than the log boundary condition at this point. In the next section we will show that at this special point an even looser boundary condition called log-square boundary condition can be consistent.

III. CONSISTENT BOUNDARY CONDITIONS

In this section, we will first derive the expression for conserved charges and then study the consistency of the Brown-Henneaux and log boundary conditions with this theory. We will also show that at the special point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$ we can have another consistent boundary condition.

A. Conserved charges

In this subsection we will give the basic formulas to calculate the conserved charges using the covariant formalism [35,47–49] (see also [50–57]) for this generalized massive gravity.

For convenience we define

$$\mathcal{G}_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}. \tag{3.1}$$

Then the covariant energy momentum tensor for the linearized gravitional excitations of this new massive gravity theory can be identified as

$$\begin{aligned}32\pi m^2 G T_{\mu\nu}=&(2m^2+5\Lambda)\mathcal{G}^{(1)}_{\mu\nu}\\ &-\frac{1}{2}(\bar{g}_{\mu\nu}\bar{\nabla}^2-\bar{\nabla}_\mu\bar{\nabla}_\nu+2\Lambda\bar{g}_{\mu\nu})R^{(1)}\\ &-2(\bar{\nabla}^2\mathcal{G}^{(1)}_{\mu\nu}-\Lambda\bar{g}_{\mu\nu}R^{(1)})+\frac{2m^2}{\mu}C^{(1)\mu\nu}.\end{aligned} \tag{3.2}$$

It is shown in [58,59] that when the background spacetime admits a Killing vector ξ_μ, the current

$$\mathcal{K}^\mu=16\pi G\xi_\nu T^{\mu\nu} \tag{3.3}$$

is covariantly conserved $\bar{\nabla}_\mu\mathcal{K}^\mu=0$. Then there exists an antisymmetric two form tensor $\mathcal{F}^{\mu\nu}$ such that

$$\mathcal{K}^\mu=16\pi G\bar{\nabla}_\nu\mathcal{F}^{\mu\nu} \tag{3.4}$$

and the corresponding charge could be written as a surface integral as

$$\begin{aligned}Q(\xi)&=-\frac{1}{8\pi G}\int_M\sqrt{-\bar{g}}\mathcal{K}^0\\ &=-\frac{1}{8\pi G}\int_{\partial M}dS_i\sqrt{-\bar{g}}\mathcal{F}^{0i},\end{aligned} \tag{3.5}$$

where ∂M is the boundary of a spacelike surface M. We have chosen M as constant time surface here and the expression is under the coordinate system of (2.8).

We can rewrite each term in the expression of the energy momentum tensor (3.2) as a total covariant derivative term [58–61] by using the definition and the properties of killing vectors:

$$\bar{\nabla}_\mu \xi^\mu = 0, \qquad \bar{\nabla}_\sigma \bar{\nabla}^\mu \xi^\sigma = 2\Lambda \xi^\mu, \qquad \bar{\nabla}^2 \xi_\sigma = -2\Lambda \xi_\sigma \tag{3.6}$$

and the final result is

$$\xi_\nu T^{\mu\nu} = \bar{\nabla}_\sigma \mathcal{F}^{\mu\sigma}, \tag{3.7}$$

where

$$\begin{aligned}\mathcal{F}^{\mu\sigma} = &\Big(1 - \frac{1}{2m^2\ell^2}\Big) F[\xi]^{\mu\sigma} + \frac{1}{\mu} F[\eta]^{\mu\sigma} \\ &- \frac{1}{4m^2}\{\xi^\mu \bar{\nabla}^\sigma R^{(1)} + R^{(1)} \bar{\nabla}^\mu \xi^\sigma - \xi^\sigma \bar{\nabla}^\mu R^{(1)}\} \\ &- \frac{1}{m^2}\{\xi_\nu \bar{\nabla}^\sigma \mathcal{G}^{(1)\mu\nu} - \xi_\nu \bar{\nabla}^\mu \mathcal{G}^{(1)\sigma\nu} - \mathcal{G}^{(1)\mu\nu} \bar{\nabla}^\sigma \xi_\nu \\ &+ \mathcal{G}^{(1)\sigma\nu} \bar{\nabla}^\mu \xi_\nu\} + \frac{1}{\mu}\{\xi_\lambda (\varepsilon^{\mu\sigma\nu} \mathcal{G}^{(1)\lambda}_\nu \\ &- \frac{1}{2}\varepsilon^{\mu\sigma\lambda} \mathcal{G}^{(1)})\}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}F[\xi]^{\mu\sigma} = &\frac{1}{2}\{\xi_\nu \bar{\nabla}^\mu h^{\sigma\nu} - \xi_\nu \bar{\nabla}^\sigma h^{\mu\nu} + \xi^\mu \bar{\nabla}^\sigma h - \xi^\sigma \bar{\nabla}^\mu h \\ &+ h^{\mu\nu} \bar{\nabla}^\sigma \xi_\nu - h^{\sigma\nu} \bar{\nabla}^\mu \xi_\nu + \xi^\sigma \bar{\nabla}_\nu h^{\mu\nu} \\ &- \xi^\mu \bar{\nabla}_\nu h^{\sigma\nu} + h \bar{\nabla}^\mu \xi^\sigma\}, \\ \eta_\mu = &\frac{1}{2}\varepsilon_{\mu\nu\lambda} \bar{\nabla}^\nu \xi^\lambda.\end{aligned} \tag{3.9}$$

Thus the conserved charge (3.5) becomes

$$Q(\xi) = -\frac{1}{8\pi G}\int_{\partial M} dS_i \sqrt{-\bar{g}} \mathcal{F}^{0i}. \tag{3.10}$$

Here we choose the spacelike surface as the constant time surface, then the expression for the conserved charge could be simplified as

$$Q(\xi) = -\lim_{\rho\to\infty} \frac{1}{8\pi G}\int d\phi \sqrt{-\bar{g}} \mathcal{F}^{0\rho}, \tag{3.11}$$

where ρ is the radial coordinate of AdS_3.

Note that here to get the formula for the conserved charges, we have used the definition of the killing vectors $\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu = 0$ which does not hold any more for asymptotic symmetries of the spacetime. Thus for asymptotic symmetries which do not obey $\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu = 0$, (3.3) is no longer a conserved quantity and we need to add some terms composed by $\bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$ and $h_{\mu\nu}$ [28]. However, the formula (3.11) will still be valid to the linearized level of gravitational excitations in our consideration.

B. Brown-Henneaux boundary condition

In this section we will analyze the Brown-Henneaux boundary condition [5] for the generalized massive gravity theory. We will calculate the conserved charges corresponding to the generators of the asymptotical symmetry under this boundary condition and see whether all the charges are finite or not. In this and the following sections we will work in the global coordinate system (2.8).

The Brown-Henneaux boundary condition for the linearized gravitational excitations in asymptotical AdS_3 spacetime can be written as

$$\begin{pmatrix} h_{++} = \mathcal{O}(1) & h_{+-} = \mathcal{O}(1) & h_{+\rho} = \mathcal{O}(e^{-2\rho}) \\ h_{-+} = h_{+-} & h_{--} = \mathcal{O}(1) & h_{-\rho} = \mathcal{O}(e^{-2\rho}) \\ h_{\rho+} = h_{+\rho} & h_{\rho-} = h_{-\rho} & h_{\rho\rho} = \mathcal{O}(e^{-2\rho}) \end{pmatrix} \tag{3.12}$$

in the global coordinate system.

The corresponding asymptotic Killing vectors are

$$\begin{aligned}\xi &= \xi^+ \partial_+ + \xi^- \partial_- + \xi^\rho \partial_\rho \\ &= [\epsilon^+(\tau^+) + 2e^{-2\rho}\partial_-^2 \epsilon^-(\tau^-) + \mathcal{O}(e^{-4\rho})]\partial_+ \\ &\quad + [\epsilon^-(\tau^-) + 2e^{-2\rho}\partial_+^2 \epsilon^+(\tau^+) + \mathcal{O}(e^{-4\rho})]\partial_- \\ &\quad - \frac{1}{2}[\partial_+ \epsilon^+(\tau^+) + \partial_- \epsilon^-(\tau^-) + \mathcal{O}(e^{-2\rho})]\partial_\rho.\end{aligned} \tag{3.13}$$

Because ϕ is periodic, we could choose the basis $\epsilon_m^+ = e^{im\tau^+}$ and $\epsilon_n^- = e^{in\tau^-}$ and denote the corresponding killing vectors as ξ_m^L and ξ_n^R. The algebra structure of these vectors is

$$\begin{aligned}i[\xi_m^L, \xi_n^L] &= (m-n)\xi_{m+n}^L, \\ i[\xi_m^R, \xi_n^R] &= (m-n)\xi_{m+n}^R, \qquad [\xi_m^L, \xi_n^R] = 0.\end{aligned} \tag{3.14}$$

Thus these asymptotic Killing vectors give two copies of Virasora algebra. To calculate the conserved charges using (3.11) we first parametrize the gravitons as follows

$$\begin{aligned}h_{++} &= f_{++}(\tau,\phi) + \ldots \\ h_{+-} &= f_{+-}(\tau,\phi) + \ldots \\ h_{+\rho} &= e^{-2\rho} f_{+\rho}(\tau,\phi) + \ldots \\ h_{--} &= f_{--}(\tau,\phi) + \ldots \\ h_{-\rho} &= e^{-2\rho} f_{-\rho}(\tau,\phi) + \ldots \\ h_{\rho\rho} &= e^{-2\rho} f_{\rho\rho}(\tau,\phi) + \ldots,\end{aligned} \tag{3.15}$$

where $f_{\mu\nu}$ depends only on τ and ϕ while not on ρ and the "..." terms are lower order terms which do not contribute to the conserved charges. After plugging (3.15) into (3.11) and performing the $\rho \to \infty$ limit, we obtain

$$Q = \frac{1}{8\pi G\ell}\int d\phi\bigg\{\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg)\epsilon^+ f_{++} + \bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\epsilon^- f_{--} - \bigg(1+\frac{1}{2m^2\ell^2}\bigg) \times \frac{(\epsilon^+ + \epsilon^-)(16f_{+-}-f_{\rho\rho})}{16}\bigg\} \tag{3.16}$$

for this theory. Three components of the equation of motion (2.13), which do not involve second derivative terms, can be viewed as asymptotic constraints. The $\rho\rho$ component gives

$$16f_{+-} - f_{\rho\rho} = 0 \tag{3.17}$$

at the boundary and the $+\rho$ and $-\rho$ components give

$$\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg)\partial_- f_{++} = \bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\partial_+ f_{--} = 0 \tag{3.18}$$

respectively. After plugging in these boundary constraints, the conserved charges become

$$Q = \frac{1}{8\pi G\ell}\int d\phi\bigg\{\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg)\epsilon^+ f_{++} + \bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\epsilon^- f_{--}\bigg\} = Q_L + Q_R, \tag{3.19}$$

where the left-moving conserved charge is

$$Q_L = \frac{1}{8\pi G\ell}\int d\phi\bigg\{\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg)\epsilon^+ f_{++}\bigg\}, \tag{3.20}$$

and the right-moving conserved charge

$$Q_R = \frac{1}{8\pi G\ell}\int d\phi\bigg\{\bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\epsilon^- f_{--}\bigg\}. \tag{3.21}$$

The left-moving and right-moving conserved charges fulfill two copies of Virasoro algebra with central charges

$$c_L = \frac{3\ell}{2G}\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg), \quad c_R = \frac{3\ell}{2G}\bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg). \tag{3.22}$$

We can see that the conserved charges Q are always finite for arbitrary value of m, μ, so the Brown-Henneaux boundary condition is always consistent with the generalized massive gravity theory. At the critical line $\frac{1}{2m^2\ell^2} + \frac{1}{\mu\ell} = 1$, the left-moving conserved charges vanish which shows that the generalized massive gravity theory is chiral at the critical line under the Brown-Henneaux boundary condition. We can see that the TMG chiral gravity at $\mu\ell = 1$ is just the special case $m^2 \to \infty$ of this generalized gravity theory.

C. Log boundary condition

As we have checked in the previous section, at the critical line $\frac{1}{2m^2\ell^2} + \frac{1}{\mu\ell} = 1$, new solutions of the equation of motion (2.13) appeared. The new solution does not obey the Brown-Henneaux boundary conditions. In order to include these new interesting solutions, the boundary conditions need to be loosened [30,33,35]. Earlier investigations on the relaxation of the boundary conditions for gravity coupled with scalar fields in anti-de Sitter spacetime could be found in [62–66].

We relax the boundary condition as follows to include the solution $\psi^{\mathrm{new}L}_{\mu\nu}$:

$$\begin{pmatrix} h_{++} = \mathcal{O}(\rho) & h_{+-} = \mathcal{O}(1) & h_{+\rho} = \mathcal{O}(\rho e^{-2\rho}) \\ h_{-+} = h_{+-} & h_{--} = \mathcal{O}(1) & h_{-\rho} = \mathcal{O}(e^{-2\rho}) \\ h_{\rho+} = h_{+\rho} & h_{\rho-} = h_{-\rho} & h_{\rho\rho} = \mathcal{O}(e^{-2\rho}) \end{pmatrix}. \tag{3.23}$$

Then the corresponding asymptotic Killing vector can be calculated to be

$$\begin{aligned} \xi &= \xi^+\partial_+ + \xi^-\partial_- + \xi^\rho\partial_\rho \\ &= [\epsilon^+(\tau^+) + 2e^{-2\rho}\partial_-^2\epsilon^-(\tau^-) + \mathcal{O}(e^{-4\rho})]\partial_+ \\ &\quad + [\epsilon^-(\tau^-) + 2e^{-2\rho}\partial_+^2\epsilon^+(\tau^+) + \mathcal{O}(\rho e^{-4\rho})]\partial_- \\ &\quad - \frac{1}{2}[\partial_+\epsilon^+(\tau^+) + \partial_-\epsilon^-(\tau^-) + \mathcal{O}(e^{-2\rho})]\partial_\rho. \end{aligned} \tag{3.24}$$

Note that these asymptotic Killing vectors are different from (3.13) only in the subleading order, so these also give two copies of Varasoro algebra the same as (3.14).

With this new boundary condition we can parametrize the asymptotic excitations as follows

$$\begin{aligned} h_{++} &= \rho f^L_{++}(\tau,\phi) + \ldots \\ h_{+-} &= f^L_{+-}(\tau,\phi) + \ldots \\ h_{+\rho} &= \rho e^{-2\rho} f^L_{+\rho}(\tau,\phi) + \ldots \\ h_{--} &= f^L_{--}(\tau,\phi) + \ldots \\ h_{-\rho} &= e^{-2\rho} f^L_{-\rho}(\tau,\phi) + \ldots \\ h_{\rho\rho} &= e^{-2\rho} f^L_{\rho\rho}(\tau,\phi) + \ldots . \end{aligned} \tag{3.25}$$

Note that $f^L_{\mu\nu}$ depends only on τ, ϕ while not on ρ and the "..." terms are subleading terms which do not contribute to the conserved charge. After plugging (3.25) into (3.11) and performing the $\rho \to \infty$ limit, we could obtain

$$Q = \frac{1}{8\pi G\ell}\int d\phi\bigg\{\bigg(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\bigg)\epsilon^+ f^L_{++}\cdot\infty + \bigg(\frac{2}{m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\epsilon^+ f^L_{++} + \bigg(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\bigg)\epsilon^- f^L_{--} - \bigg(1+\frac{1}{2m^2\ell^2}\bigg)\frac{(\epsilon^+ + \epsilon^-)(16f^L_{+-}-f^L_{\rho\rho})}{16}\bigg\}. \tag{3.26}$$

The first term is a linear divergent term proportional to ρ at infinity, which is caused by the relaxation of the boundary condition. We see that the conserved charges can only be finite at the critical line $\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}=1$, which means that the log boundary condition is only well-defined at the line $\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}=1$. The asymptotic constraints coming from the equation of motion (2.13), which are

$$16f^L_{+-}-f^L_{\rho\rho}=0, \tag{3.27}$$

and

$$\left(\frac{2}{m^2\ell^2}+\frac{1}{\mu\ell}\right)\partial_-f^L_{++}=\partial_+f^L_{--}=0. \tag{3.28}$$

Now at the critical line $\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}=1$, the conserved charges become

$$\begin{aligned} Q_L&=\frac{1}{8\pi G\ell}\int d\phi\left[\left(1+\frac{3}{2m^2\ell^2}\right)\epsilon^+f^L_{++}\right],\\ Q_R&=\frac{1}{4\pi G\ell}\int d\phi\left[\frac{1}{\mu\ell}\epsilon^-f^L_{--}\right]. \end{aligned} \tag{3.29}$$

Thus we have loosened the boundary condition to get nonzero left-moving charges. The central charges are not altered with the same argument for TMG in [35].

Note that at the special point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$, $Q_L=0$ under this boundary condition which means that the theory is still chiral and there is the possibility that we can further relax the boundary condition to have nonzero left-moving conserved charges. This coincides with the fact that at this special point new solutions which do not obey the log boundary conditions emerge. In the next subsection we will give the new boundary condition at this special point.

D. Log-Square boundary condition

At the critical point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$, a kind of new solution (2.29) appeared. This new solution does not obey either the Brown-Henneaux boundary conditions or the log boundary condition, and we have to relax the boundary condition again to accommodate such kind of solutions.

With the hint from the asymptotical behavior of the new solution (2.29) we can relax the boundary condition as follows to include the solution $\psi^{\mathrm{new}S}_{\mu\nu}$

$$\begin{pmatrix} h_{++}=\mathcal{O}(\rho^2) & h_{+-}=\mathcal{O}(1) & h_{+\rho}=\mathcal{O}(\rho^2e^{-2\rho})\\ h_{-+}=h_{+-} & h_{--}=\mathcal{O}(1) & h_{-\rho}=\mathcal{O}(e^{-2\rho})\\ h_{\rho+}=h_{+\rho} & h_{\rho-}=h_{-\rho} & h_{\rho\rho}=\mathcal{O}(e^{-2\rho}) \end{pmatrix}. \tag{3.30}$$

Then the corresponding asymptotic Killing vector can be calculated to be

$$\begin{aligned} \xi&=\xi^+\partial_++\xi^-\partial_-+\xi^\rho\partial_\rho\\ &=[\epsilon^+(\tau^+)+2e^{-2\rho}\partial_-^2\epsilon^-(\tau^-)+\mathcal{O}(e^{-4\rho})]\partial_+\\ &\quad+[\epsilon^-(\tau^-)+2e^{-2\rho}\partial_+^2\epsilon^+(\tau^+)+\mathcal{O}(\rho^2e^{-4\rho})]\partial_-\\ &\quad-\frac{1}{2}[\partial_+\epsilon^+(\tau^+)+\partial_-\epsilon^-(\tau^-)+\mathcal{O}(e^{-2\rho})]\partial_\rho. \end{aligned} \tag{3.31}$$

Note that these asymptotic Killing vectors are different from (3.13) only in the subleading order, so these also give two copies of Varasoro algebra the same as (3.14).

With this new boundary condition we can parametrize the asymptotic excitations as follows

$$\begin{aligned} h_{++}&=\rho^2f^S_{++}(\tau,\phi)+\ldots\\ h_{+-}&=f^S_{+-}(\tau,\phi)+\ldots\\ h_{+\rho}&=\rho^2e^{-2\rho}f^S_{+\rho}(\tau,\phi)+\ldots\\ h_{--}&=f^S_{--}(\tau,\phi)+\ldots\\ h_{-\rho}&=e^{-2\rho}f^S_{-\rho}(\tau,\phi)+\ldots\\ h_{\rho\rho}&=e^{-2\rho}f^S_{\rho\rho}(\tau,\phi)+\ldots. \end{aligned} \tag{3.32}$$

Note that $f^S_{\mu\nu}$ depends only on τ, ϕ while not on ρ and the "…" terms are subleading terms which do not contribute to the conserved charge. After plugging (3.32) into (3.11) and performing the $\rho\to\infty$ limit, we could obtain

$$\begin{aligned} Q=\frac{1}{8\pi G\ell}\int d\phi\bigg\{&\left(1-\frac{1}{2m^2\ell^2}-\frac{1}{\mu\ell}\right)\epsilon^+f^S_{++}\cdot[\lim_{\rho\to\infty}\rho^2]\\ &-\left(1-\frac{9}{2m^2\ell^2}-\frac{3}{\mu\ell}\right)\epsilon^+f^S_{++}\cdot[\lim_{\rho\to\infty}\rho]\\ &-\left(\frac{4}{m^2\ell^2}+\frac{1}{\mu\ell}\right)\epsilon^+f^S_{++}+\left(1-\frac{1}{2m^2\ell^2}+\frac{1}{\mu\ell}\right)\epsilon^-f^S_{--}\\ &-\left(1+\frac{1}{2m^2\ell^2}\right)\frac{(\epsilon^++\epsilon^-)(16f^S_{+-}-f^S_{\rho\rho})}{16}\bigg\}. \end{aligned} \tag{3.33}$$

Note that we did not include the next to leading order term in the asymptotic behavior (3.32), i.e. the term proportional to ρ in h_{++}. This does not affect our result because the conserved charges are linear in $h_{\mu\nu}$. We see that the conserved charges can only be finite at the critical point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$, which means that this log-square boundary condition is only well-defined at the special point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$. The asymptotic constraints coming from the equation of motion (2.13) are

$$16f^S_{+-}-f^S_{\rho\rho}=0, \tag{3.34}$$

and

$$\partial_-f^S_{++}=\partial_+f^S_{--}=0 \tag{3.35}$$

Now at the point $m^2\ell^2=-2\mu\ell=-\frac{3}{2}$, the conserved charges become

$$Q_L = \frac{1}{6\pi G\ell}\int d\phi[\epsilon^+ f^S_{++}],$$
$$Q_R = \frac{1}{3\pi G\ell}\int d\phi[\epsilon^- f^S_{--}]. \tag{3.36}$$

The central charges are not changed because the Lie derivative of f^S_{++} also involves no terms like $\partial_+\epsilon^+ + \partial^3_+\epsilon^+$ which contributes to the central charge.

IV. CONCLUSION AND DISCUSSION

In this note we have studied the generalized massive gravity in asymptotically AdS_3 spacetime by combining the two mass terms of TMG and NMG. This generalized massive gravity no longer possesses the left-right symmetry because of the addition of the Chern-Simons term. Thus in this theory there exists a chiral line $c_L = 0$ at which the theory becomes chiral and only right-moving modes exist. The conserved charges are calculated to show that the left moving conserved charges indeed vanish at $c_L = 0$ under Brown-Henneaux boundary condition.

At the linearized level, we calculated the highest weight graviton solutions of this massive gravity. Very similar to the case of TMG, at $c_L = 0$ one massive graviton coincides with the left-moving massless mode and a new solution with log asymptotic behavior appeared. Thus at the critical line $c_L = 0$ the boundary condition can be loosened to the log boundary condition and we showed that this boundary condition is indeed consistent with the theory at $c_L = 0$. However, because we have two adjustable mass parameters in this generalized massive gravity, a new special point emerged which does not exist in TMG or NMG. At this special point, both of the massive graviton solutions become left-moving massless modes and we can have another new solution besides the one which also exists in TMG and NMG. This new solution has a log-square asymptotic behavior and we showed that a new boundary condition called log-square boundary condition which can accommodate this new solution can be consistent with the theory at this special point.

It has been suggested in [15,22,29] that the logarithmic CFT may be the dual field theory of TMG at the chiral point with the log boundary condition. It would be a challenge if we can find the dual field theory which has similar properties with this GMG at the special point where the log-square boundary condition can be imposed.

It is interesting to find whether there are classical solutions that exhibit the log-square asymptotic behavior, like the solutions in [20,23,31,67,68] for TMG. Also it would be interesting to study other consistent boundary conditions for this generalized massive gravity in asymptotically AdS_3 at both the critical line and other value of the mass parameters. To gain further understanding of the dual field theory of this generalized massive gravity, we need to study the classical solutions of this theory, which contribute to the sum over geometry. The entropy of the BTZ black hole calculated using the central charges obtained under Brown-Henneaux or log (square) boundary conditions is the same to the one calculated using other methods, e.g. in [44]. This is also the case for NMG [40–42]. We should further study this subject to see if there can be other black hole solutions under these boundary conditions.

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