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FINITE-SIZE SCALING OF THE CORRELATION LENGTH IN ANISOTROPIC SYSTEMS*

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The finite-size scaling functions of thermodynamic functions in anisotropic systems have been shown to be dependent on the spatial anisotropy [X.S. Chen and V. Dohm, Phys. Rev. E 70, 056136 (2004)]. Here we extend this study to the correlation length $\xi_{||}$ of the anisotropic O(n) symmetric φ^4 model in an $L^{d-1} \times \infty$ cylindric geometry with periodic boundary conditions. We calculate the exact finite-size scaling function of correlation length $\xi_{||}$ for $T \geq T_c$ in $2 < d < 4$ dimensions and in the limit $n \to \infty$. The finite-size scaling function of $\xi_{||}$ is dependent on a normalized symmetric $(d-1) \times (d-1)$ matrix defined by the anisotropy matrix of anisotropic systems .

Keywords: Critical phenomena; finite-size scaling; anisotropic systems.

1. Introduction

The finite-size scaling ansatz in the critical phenomena of confined systems was introduced by M. E. Fisher.[1] It has been applied widely in the different parts of physics.[2] The notion of universality is of fundamental importance to the physics of critical phenomena, both in bulk and in confined systems. For d-dimensional systems with an n-component order parameter, the asymptotic *bulk* critical behavior is characterized by universal quantities (critical exponents, amplitude ratios and scaling functions) that are independent of microscopic details. The universality is classified by (d, n). (See, e.g.,the review article.[2]) The universality of the finite-size scaling[1] in confined systems has been hypothesized,[3] except that the finite-size scaling functions depend on the geometry and boundary conditions.

Specifically, it was predicted that the singular part of the reduced free energy density of a confined system with characteristic size L had the asymptotic (near

*Dedicated to Prof. Bambi Hu on the occasion of his 60th birthday.

bulk critical temperature T_c, small ordering field h, large L) universal scaling form[3]

$$f_s(t, h, L) = L^{-d} Y(C_1 t L^{1/\nu}, C_2 h L^{\beta\delta/\nu}), \tag{1}$$

where $t = (T - T_c)/T_c$ is the reduced temperature. The finite-size scaling function $Y(x, y)$ was supposed to be universal and depends on only the dimensionality d of the system and the number n of the components of the order parameter, apart from the dependence on the geometry and on the boundary conditions. Thus the two factors C_1 and C_2 were predicted to be the only nonuniversal parameters that depend on the microscopic properties of the system. This is parallel to the hypothesis of two-scale factor universality in the bulk critical phenomena.[2] While other thermodynamic quantities of confined systems can be derived from (1), correlation lengths such as $\xi_{||}$ are independent quantities to be derived from correlation function or from subdominant eigenvalues of transfer matrix.[3,4] An analogous hypothesis was made for the correlation length $\xi_{||}(t, h, L)$ in an $L^{d-1} \times \infty$ cylinder[3]

$$\xi_{||}(t, h, L) = L X(C_1 t L^{1/\nu}, C_2 h L^{\beta\delta/\nu}), \tag{2}$$

with an independent universal scaling function $X(x, y)$ and with the same nonuniversal parameters C_1 and C_2 as in (1). The correlation length $\xi_{||}$ constitutes the basic length scale of the order-parameter correlation function in the longitudinal direction of the cylinder.

One possible source of nonuniversality is the lattice anisotropy which is known to be a marginal perturbation in the renormalization-group sense, thus a dependence of the asymptotic critical behavior on anisotropy parameters cannot be excluded *a priori*. For example, it is well known that there exist different bulk correlation length amplitudes in the different directions of the principle axes of an anisotropic system. It was widely believed, however, that such anisotropy can be eliminated by an anisotropic transformation of length scales[3,5,6,7,8,9,10,11,12,13,14,15] which then restores isotropy and thus relates the asymptotic critical behavior of anisotropic systems to the universal critical behavior of isotropic systems within a given (d, n) universality class.

It has been demonstrated[16] that the universal scaling form (1) is not valid for general anisotropic systems. In the present paper we will specify the finite-size scaling form of the correlation length $\xi_{||}$ by the exact result of $X = \xi_{||}(t, L; \mathbf{A})/L$ in $2 < d < 4$ dimensions and in the limit $n \to \infty$. Exact results in the large-n limit have been derived previously by Brézin[17] for the case of a simple-cubic lattice with *isotropic* nearest neighbor interactions corresponding to $\mathbf{A} = \mathbf{1}$. For *isotropic* field-theoretic models, $\xi_{||}$ has also been calculated for finite n within the $\epsilon = 4 - d$ expansion.[18] Here we study the case $n \to \infty$ with a *nondiagonal* anisotropy matrix $\mathbf{A}$. Instead of the universal scaling form in Eq.(2) we shall find that the correlation length $\xi_{||}$ has a *nonuniversal* scaling form with an intrinsic dependence on the anisotropic matrix $\bar{\mathbf{A}}$ that cannot be eliminated by a transformation of the length L.

Most of the previous studies of lattice models were focussed on the amplitude

$$X(0,0) = \xi_{||}(0,0,L)/L \tag{3}$$

at the bulk critical point $t = 0$ and $h = 0$, as reviewed in [2,19], since this amplitude was predicted to be universal.[3,4] In most cases there was no anisotropy effect on $X(0,0)$ because of the cubic symmetry of the underlying models. A few of these studies were performed for models with anisotropic nearest-neighbor couplings on simple-cubic lattices[4,12] or with isotropic nearest-neighbor couplings on a noncubic lattice.[3] In these cases the anisotropy effect of $X(0,0)$ could be absorbed completely by a rescaling of lattice spacings or of the length L.

2. Correlation Length

Within the spatially anisotropic O(n) symmetric φ^4 Hamiltonian

$$H(r_0, u_0, \Lambda; \mathbf{A}; V; \varphi) = \int_V d^d x \left[\frac{r_0}{2}\varphi^2 + \sum_{\alpha,\beta}^{d} \frac{A_{\alpha\beta}}{2} \frac{\partial\varphi}{\partial x_\alpha} \frac{\partial\varphi}{\partial x_\beta} + u_0(\varphi^2)^2 \right] \tag{4}$$

for the n-component order parameter $\varphi(\mathbf{x})$ in $2 < d < 4$ dimensions, we shall show that the amplitude in Eq.(3) is nonuniversal for the nondiagonal matrix $\mathbf{A}$ in the limit $n \to \infty$, even after a rescaling of lengths. For simplicity we consider a $L^{d-1} \times \infty$ cylinder with periodic boundary conditions in the $d-1$ horizontal directions. The $d-1$ horizontal and vertical coordinates are denoted by $\mathbf{y}$ and z, respectively, with $\mathbf{x} = (\mathbf{y}, z)$.

The order-parameter correlation function (divided by n) of the confined system for $T \geq T_c$ and $h = 0$ is defined as

$$G(\mathbf{y}, z) = \frac{1}{n} < \varphi(\mathbf{x})\varphi(0) > \tag{5}$$

where $< \ldots >$ denotes an average with the statistical weight $\propto \exp(-H)$. The field $\varphi(\mathbf{x})$ can be represented as $\varphi(\mathbf{x}) = L^{-(d-1)} \sum_{\mathbf{q}} \int_p \varphi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$, where the Fourier components $\varphi_{\mathbf{k}} = \int_V d^d\mathbf{x}\, \varphi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = (\mathbf{q}, p)$. The sum $\sum_{\mathbf{q}}$ runs over $(d-1)$-dimensional $\mathbf{q}$ vectors with components $q_\alpha = 2\pi m_\alpha / L, m_\alpha = 0, \pm 1, \pm 2, \ldots$ up to some cutoff Λ.

Parallel to the isotropic case [20], in the limit $n \to \infty$ at fixed $u_0 n$ we have

$$G(\mathbf{y}, z) = L^{-(d-1)} \sum_{\mathbf{q}} \int_p \left[\hat{G}(\mathbf{0})^{-1} + \mathbf{k}\cdot\mathbf{A}\cdot\mathbf{k} \right]^{-1} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6}$$

with $\mathbf{k}\cdot\mathbf{A}\cdot\mathbf{k} = \sum_{\alpha,\beta}^{d} A_{\alpha\beta} k_\alpha k_\beta$. The susceptibility $\hat{G}(\mathbf{0})$ is determined implicitly by

$$\hat{G}(0)^{-1} = r_0 + 4u_0 n L^{-(d-1)} \sum_{\mathbf{q}} \int_p \left[\hat{G}(0)^{-1} + \mathbf{k}\cdot\mathbf{A}\cdot\mathbf{k} \right]^{-1}. \tag{7}$$

In the following we omit the effects related to the cutoff Λ and take the limit $\Lambda \to \infty$. Using the Poisson formula[21] Eq. (6) becomes

$$G(\mathbf{y}, z) = \sum_{\mathbf{m}} \int_{\mathbf{k}} \left[\hat{G}(\mathbf{0})^{-1} + \mathbf{k}\cdot\mathbf{A}\cdot\mathbf{k}\right]^{-1} e^{i\mathbf{k}\cdot\mathbf{x}_{\mathbf{m}L}} \tag{8}$$

with $\mathbf{x}_{\mathbf{m}L} = (\mathbf{y} + \mathbf{m}L, z)$. After the integration we have

$$G(\mathbf{y}, z) = (\det \mathbf{A})^{-1/2} \sum_{\mathbf{m}} \frac{\Phi(|\mathbf{x}'_{\mathbf{m}L}|/\hat{G}(\mathbf{0})^{1/2})}{|\mathbf{x}'_{\mathbf{m}L}|^{d-2}} \tag{9}$$

with $|\mathbf{x}'_{\mathbf{m}L}| \equiv \sqrt{\mathbf{x}_{\mathbf{m}L}\cdot\mathbf{A}^{-1}\cdot\mathbf{x}_{\mathbf{m}L}}$ and $\Phi(x) = (2\pi)^{-d/2}x^{(d-2)/2}K_{(d-2)/2}(x)$ where $K_\nu(x)$ is the modified Bessel function.[22] From the large-x behavior of $K_\nu(x)$[23] we obtain the asymptotic behavior

$$G(\mathbf{y}, z) \propto \sum_{\mathbf{m}} \frac{e^{-|\mathbf{x}'_{\mathbf{m}L}|\hat{G}(\mathbf{0})^{-1/2}}}{|\mathbf{x}'_{\mathbf{m}L}|^{(d-1)/2}}, \quad |\mathbf{x}| \gg \hat{G}(\mathbf{0})^{1/2}. \tag{10}$$

Finally we obtain the correlation length $\xi_{||}$ as

$$\xi_{||}(t, L)^{-1} = -\lim_{|z|\to\infty} \frac{1}{|z|}\log G(\mathbf{0}, z) = \left[\hat{G}(\mathbf{0})/(\mathbf{A}^{-1})_{dd}\right]^{-1/2}. \tag{11}$$

In the asymptotic region with $L \gg \Lambda^{-1}$ and $0 \le t \ll 1$, From Eqs.(7) and (11) we obtain an asymptotic finite-size scaling form of the correlation length

$$\xi_{||}(t, L; \mathbf{A}) = L'/[(\mathbf{A}^{-1})_{dd}]^{1/2} f(L'/\xi'_{bulk}; \bar{\mathbf{B}}), \tag{12}$$

where $L' = L/(\det \mathbf{B})^{1/2(d-1)}$ and the bulk correlation length $\xi'_{bulk} = \xi'_0(\mathbf{A})t^{-\nu}$ with $\nu = 1/(d-2)$[16]. The $(d-1)\times(d-1)$ matrix $\mathbf{B}$ is defined as $B_{\alpha\beta} = A_{\alpha\beta} - A_{\alpha d}A_{d\beta}/A_{dd}$, $\alpha, \beta = 1, 2, ...d-1$ and $\bar{\mathbf{B}} = \mathbf{B}/(\det \mathbf{B})^{1/(d-1)}$. The scaling function $f(L'/\xi'_{bulk}; \bar{\mathbf{B}})$ is determined implicitly by

$$f^{2-d} = (L'/\xi'_{bulk})^{1/\nu} - \frac{4-d}{4\pi^2 A_d}\int_0^\infty ds(\pi/s)^{1/2}P(s, \bar{\mathbf{B}})e^{-f^{-2}s/4\pi^2}, \tag{13}$$

where $A_d = \Gamma(3 - d/2)2^{2-d}\pi^{-d/2}(d-2)^{-1}$ and $P(s, \bar{\mathbf{B}}) = (\pi/s)^{(d-1)/2} - \sum_{\mathbf{m}} e^{-\mathbf{m}\cdot\bar{\mathbf{B}}\cdot\mathbf{m}s}$. The universal finite-size scaling form (2) is now replaced by the *nonuniversal* scaling form

$$\xi_{||}(t, L; \mathbf{A}) = LX(C_1 t L^{1/\nu}; \bar{\mathbf{A}}) = L\left[\bar{A}_{dd}^{1/(d-1)}/(\bar{\mathbf{A}}^{-1})_{dd}\right]^{1/2} f(L'/\xi'_{bulk}; \bar{\mathbf{B}}). \tag{14}$$

The scaling functions $f(L'/\xi'_{bulk}; \bar{\mathbf{B}})$ depends on the nonuniversal matrix $\bar{\mathbf{B}}$ in a highly complicated way due to the function $P(s, \bar{\mathbf{B}})$. Although the nonuniversal prefactor $\left[\bar{A}_{dd}^{1/(d-1)}/(\bar{\mathbf{A}}^{-1})_{dd}\right]^{1/2}$ could be formally adsorbed by introducing the rescaled length L'' and by rewriting $C_1 t L^{1/\nu} = C''_1 t L''^{1/\nu}$ with a different constant C''_1, the dependence on $\bar{\mathbf{B}}$ cannot be eliminated. Thus the isotropy cannot be restored by an anisotropic scale transformation if the matrix $\bar{\mathbf{B}}$ is *nondiagonal.* Such a transformation is possible in the case of a *diagonal* matrix $\mathbf{A}$ with different diagonal elements, as noted before[16] for the free energy density.

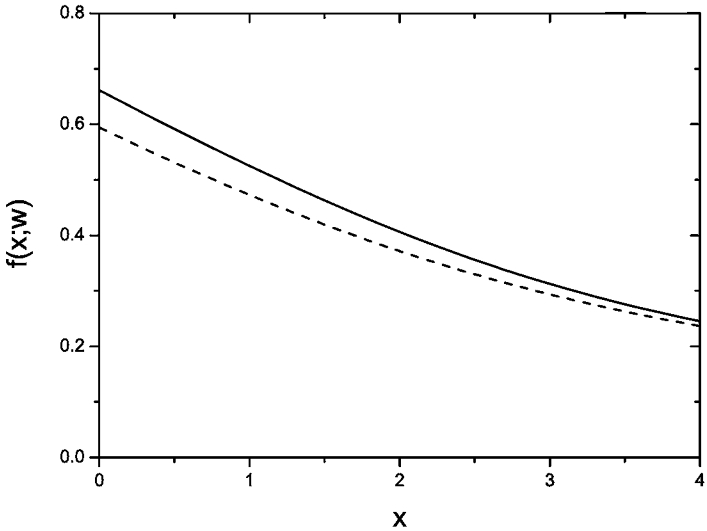

Fig. 1. finite-size scaling function $f(x;w)$ at $T \geq T_c$ with $x = L'/\xi'_{bulk} \geq 0$ in the three-dimensional model, Eq. (4), with the coupling ratio $w = J'/(J+2J') = -0.48$ (dashed line) and $w = 0$ (solid line).

3. Results

To illustrate quantitatively the nonuniversality of the finite-size scaling function $f(L'/\xi'_{bulk}; \bar{\mathbf{B}})$ of the correlation length $\xi_{||}$, we investigate a three-dimensional anisotropic φ^4 continuum model (4) with the anisotropy matrix $\mathbf{A}(w)$ as we introduced recently[16]with $A_{11} = A_{22} = A_{33} = c_0$ and $A_{12} = A_{13} = A_{23} = c_0 w$. The parameters w and c_0 are related to the isotropic nearest-neighbor interaction J and the anisotropic next-nearest-neighbor interaction J' of an anisotropic lattice φ^4 model[16] according to $w = J'/(J+2J') \leq \frac{1}{2}$ and $c_0 = 2(J+2J') > 0$. We will consider w in the range $-\frac{1}{2} < w \leq \frac{1}{2}$ where ferromagnetic critical behavior exists. From $\mathbf{A}$ we can obtain $\bar{\mathbf{B}}$ with $\bar{B}_{11} = \bar{B}_{22} = (1+w)/\sqrt{1+2w}$ and $\bar{B}_{12} = w/\sqrt{1+2w}$.

The correlation length $\xi_{||}$ of this system in an $L \times L \times \infty$ cylinder has the following finite-size scaling form

$$\xi_{||}(t, L; w) = L\alpha(w) f(L'/\xi'_{bulk}; w) \tag{15}$$

where $\alpha(w) = \left[\bar{A}_{33}^{1/2}/(\bar{\mathbf{A}}^{-1})_{33}\right]^{1/2} = (1+2w)^{1/4}(1+w)^{-1}$. The finite-size scaling functions $f(x;w)$ for $w = -0.48$ and at $T \geq T_c$ and $w = 0$ are shown in Fig.1.

4. Conclusions

In conclusion, we have investigate the finite-size scaling behavior of the correlation length $\xi_{||}$ in an $L^{d-1} \times \infty$ cylindric geometry in dimensions $2 < d < 4$ and with periodic boundary conditions. We have calculated the finite-size scaling function of $\xi_{||}$ in the spatially anisotropic O(n) symmetric φ^4 continuum model in the limit $n \to \infty$. We have demonstrated that the two-scale factor universality hypothesized by Privman and Fisher[3] is absent in the finite-size scaling function of $\xi_{||}$ for the spatially anisotropic systems. The correlation length $\xi_{||}$ of an anisotropic system characterized by the matrix $\mathbf{A}$ has a finite-size scaling form $\xi_{||}(t, L; \mathbf{A}) = L[\bar{A}_{dd}^{1/(d-1)}/(\bar{\mathbf{A}}^{-1})_{dd}]^{1/2} f(L'/\xi'_{bulk}; \bar{\mathbf{B}})$. The scaling function $f(x; \mathbf{B})$ is nonuniversal and depends on the $(d-1) \times (d-1)$ normalized anisotropy matrix $\bar{\mathbf{B}}$ in a highly complicated way. The intrinsic dependence on $\bar{\mathbf{B}}$ of the scaling function $f(x; \bar{\mathbf{B}})$ cannot be eliminated by an anisotropic scale transformation.

Although the finite-size scaling form of the correlation length $\xi_{||}$ in Eq.(14) is obtained from the anisotropic O(n) symmetric φ^4 model in the limit $n \to \infty$, we expect that this scaling form is valid for systems with finite n.

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References

1. M.E. Fisher, in *Critical Phenomena, Proc. 1970 E. Fermi Int. School of Physics*, edited by M.S. Green (Academic, New York, 1971), Vol. 51, p. 1.
2. V. Privman, A. Aharony, and P.C. Hohenberg, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, New York, 1991), Vol. 14, p. 1.
3. V. Privman and M.E. Fisher, *Phys. Rev.* **B 30**, 322 (1984).
4. P. Nightingale and H. Blöte, *J. Phys. A: Math. Gen.* **16**, L657 (1983).
5. J.L. Cardy, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, New York, 1987), Vol. 11, p. 55.
6. J.L. Cardy, *J. Phys. A: Math. Gen.* **16**, 3617 (1983).
7. M.N. Barber, I. Peschel, and P.A. Pearce, *J. Stat. Phys.* **37**, 497 (1984).
8. J.O. Indekeu, M.P. Nightingale, and W.V. Wang, *Phys. Rev.* **B 34**, 330 (1986).
9. K. Binder and J.S. Wang, *J. Stat. Phys.* **55**, 87 (1989).
10. D. Kim and P.A. Pearce, *J. Phys. A: Math. Gen.* **20**, L451 (1987).
11. G. Kamieniarz and H.W.J. Blöte, **J. Phys. A: Math. Gen. 26**, 201 (1993).
12. M.A. Yurishchev, *Phys. Rev.* **E 55**, 3915 (1997).
13. D. Karevski, P. Lajkó, and L. Turban, *J. Stat. Phys.* **86**, 1153 (1997).
14. M. Campostrini, A. Pelisetto, P. Rossi, and E. Vicari, *Phys. Rev.* **E 57**, 184 (1998).
15. A. Hucht, *J. Phys. A: Math. Gen.* **35**, L481 (2002).
16. X.S. Chen and V. Dohm, *Phys. Rev.* **E 70**, 056136 (2004); **E 71**, 059901(E) (2005).
17. E. Brézin, *J. Physique* **43**, 15 (1982).
18. E. Brézin and J. Zinn-Justin, *Nucl. Phys.* **B 257[FS14]**, 867 (1985).
19. M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer, Heidelberg, 1999).
20. X.S. Chen and V. Dohm, *Eur. Phys. J.* **B 5**, 529 (1998).
21. P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (Mc Graw-Hill, New York, 1953).

22. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
23. M. Abramowitz and I. A. Stegun, eds.,*Handbook of Mathematical Functions*(Dover Publ., New York, 1972).
24. X.S. Chen and V. Dohm, *Eur. Phys. J.* **B 15**, 283 (2000).

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