

## PARAMETRIZATION OF $k$ -ESSENCE AND ITS KINETIC TERM

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We construct the non-canonical kinetic term of a  $k$ -essence field directly from the effective equation of state function  $w_k(z)$ , which describes the properties of the dark energy. Adopting the usual parametrizations of equation of state, we numerically reproduce the shape of the non-canonical kinetic term and discuss some features of the constructed form of  $k$ -essence.

*Keywords:* Parametrization; equation of state; reconstruction;  $k$ -essence; non-canonical kinetic term.

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Recent observations of type Ia supernovae,<sup>1–5</sup> measurements of the cosmic microwave background<sup>6</sup> and the galaxy power spectrum<sup>7</sup> indicate the existence of the dark energy — an energy component with a strongly negative equation of state (EOS)  $w_{DE}$  — in addition to matter. For this purpose, various dynamical scalar field models have been proposed. Specifically, a reliable model should explain why the present amount of the dark energy is so small compared with the fundamental scale (fine-tuning problem) and why it is comparable with the critical density today (coincidence problem). Quintessence has been proposed as a candidate for the dark energy component of the universe that would be responsible of the currently observed accelerated expansion.<sup>8–12</sup> Generally speaking, quintessence is a spatially homogeneous field slow-rolling down its potential. More recently, models based on scalar fields with non-canonical kinetic energy, dubbed as  $k$ -essence, have emerged.<sup>13–17</sup> A subclass of models features a tracker behavior during radiation domination, and a cosmological-constant-like behavior shortly after the transition to matter domination. As long as this transition seems to occur generically for purely dynamical reasons, these models are claimed to solve the coincidence problem without fine-tuning. What is more, lately it has been found that those two

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theoretical setups are strongly related to each other and every quintessence model can be viewed as a  $k$ -essence model generated by a kinetic linear function.<sup>18</sup>

The dark energy is characterized by its equation of state parameter  $w_{\text{DE}}$ , which is in general a function of redshift  $z$  in dark energy models. One could specify a Lagrangian and solve the scalar field equation for the particular theory.<sup>19</sup> Although it may provide a specific form of  $w_{\text{DE}}(z)$  in terms of scalar field model parameters which can be directly constrained through data fitting,<sup>20,21</sup> this physically motivated approach clearly limits us to a model by model test.<sup>22</sup> Apart from the feasibility of quintessence potential  $V(\phi)$  and the EOS  $w_{\text{DE}}(z)$  reconstruction from supernova observations,<sup>23–27</sup> a phenomenological parametrization of  $w_{\text{DE}}(z)$  is still necessary to compare different dark energy models. The key point of this alternative strategy is to assume a more or less arbitrary ansatz for  $w_{\text{DE}}(z)$  which is not necessarily physically justified but specially designed to give a good fit to the observational data. This motivation stimulates several parametrization methods to produce an appropriate parameter space.<sup>28–33</sup> Recent study found that it is also workable to drive these parametrizations to construct the quintessence potential form<sup>34</sup> and some general features of these constructions were discussed. Here, we try to carry out this method by constructing the  $k$ -essence with pressure  $p(\varphi, X)$  directly from the dark energy equation of state function  $w_k(z)$ . When applying this method to four typical parametrizations, the shapes of the kinetic term are numerically obtained and relevant discussions on the resulting construction are also given.

Restricting our attention to a single field model, the action of the  $k$ -essence may generically be expressed as

$$S_\varphi = \int d^4x \sqrt{-g} \left[ -\frac{R}{6} + p(\varphi, X) \right], \quad (1)$$

where we use units such that  $8\pi G/3 \equiv 1$  and

$$X = \frac{1}{2}(\nabla\varphi)^2. \quad (2)$$

The Lagrangian  $p$  depends on the specific particle theory model. Here, we consider only factorizable Lagrangians of the form<sup>35</sup>

$$p = K(\varphi)\tilde{p}(X), \quad (3)$$

where we assume that  $K(\varphi) > 0$ . To describe the behavior of the scalar field it is convenient to use a perfect fluid analogy. The role of the pressure is played by the Lagrangian  $p$  itself, while the energy density is given by<sup>36</sup>

$$\begin{aligned} \varepsilon &= K(\varphi)(2X\tilde{p}_{,X}(X) - \tilde{p}(X)) \\ &\equiv K(\varphi)\tilde{\varepsilon}(X), \end{aligned} \quad (4)$$

where  $\dots_X$  denotes a partial derivative with respect to  $X$ . The ratio of pressure to energy density, which we call for brevity, the  $k$ -essence equation-of-state,

$$w_k \equiv \frac{p}{\varepsilon} = \frac{\tilde{p}}{\tilde{\varepsilon}} = \frac{\tilde{p}}{2X\tilde{p}_{,X} - \tilde{p}}, \tag{5}$$

does not depend on the function  $K(\varphi)$ .

We will consider a spatially flat FRW universe which is dominated by the nonrelativistic matter and a spatially homogeneous scalar field  $\varphi$ . The Friedmann equation can be written as

$$H^2 = \rho_m + \varepsilon, \tag{6}$$

where  $\rho_m$  is the matter density. The evolution of  $k$ -essence field is governed by the equation of motion

$$\dot{\varepsilon} + 3H(\varepsilon + p) = 0, \tag{7}$$

which yields

$$\begin{aligned} \varepsilon(z) &= \varepsilon_0 \exp \left[ 3 \int_0^z (1 + w_k) d \ln(1 + z) \right] \\ &\equiv \varepsilon_0 E(z), \end{aligned} \tag{8}$$

where  $z$  is the redshift given by  $1 + z = a_0/a$  and subscript 0 denotes the value of a quantity at the redshift  $z = 0$  (present). As the constant  $\varepsilon_0$  is of no importance for further discussion, we will safely set it to 1 and then regard the energy density  $\varepsilon(z)$  as a dimensionless quantity.

So we get  $\varepsilon(z)$  as the function of redshift  $z$ , and by means of the EOS of  $k$ -essence  $\tilde{p}(z)$  may be well prepared as follows:

$$\tilde{p}(z) = \frac{w_k(z)E(z)}{K(\varphi)}. \tag{9}$$

Once the form of the potential function  $K(\varphi)$  is specified,  $\varphi(z)$  supplied by Eq. (13) will describe the behavior of  $\tilde{p}(z)$ . With the help of Eqs. (4) and (5),  $X$  can also be implicitly expressed as the function of  $z$ :

$$\varepsilon(z) = K(\varphi) \left[ 2X \frac{d\tilde{p}(z)}{dz} \frac{dz}{dX} - \tilde{p}(z) \right]. \tag{10}$$

Therefore, the redshift relates two functions  $p(z)$  and  $X(z)$ , and the reconstruction may be implemented. Using  $\rho_m = \rho_{m0}(1 + z)^3$  and Eq. (8), the Friedmann Eq. (6) becomes

$$H(z) = H_0 [\Omega_{m0}(1 + z)^3 + \Omega_{k0}E(z)]^{1/2}, \tag{11}$$

where  $\Omega_{m0} \equiv \rho_{m0}/H_0^2$  and  $\Omega_{k0} \equiv \rho_{k0}/H_0^2$ . Due to the homogeneity of  $k$ -essence  $X = \dot{\varphi}^2/2$  and therefore,

$$X(z) = \frac{(\varphi'(z))^2(1 + z)^2 H(z)^2}{2}, \tag{12}$$

where the dot represents derivative with respect to the physical time  $t$  and the prime derivative with respect to the redshift  $z$ . As a consequence,  $\dot{\varphi} \equiv \varphi'(z)dz/dt = \pm\sqrt{2X}$ . As  $\tilde{p}(x)$  is only dependent on the variable  $X$ , the sign of  $\dot{\varphi}$  is insignificant for the construction process below.

For definiteness we will choose  $K(\varphi) = \varphi^{-2}$ .<sup>35</sup> Since  $dz/dt = -(1+z)H(z)$  and  $H(z)$  is given by Eq. (6), after straightforward calculations the differential equation for  $\varphi(z)$  will be written as the Bernoulli type

$$v'(z) + \left[ \frac{1}{1+z} + \frac{H'(z)}{H(z)} - \frac{w'_k}{1+w_k} - \frac{w_k E'(z)}{(1+w_k)E(z)} \right] v(z) - \frac{w_k - 1}{w_k + 1} v(z)^2 = 0, \quad (13)$$

where  $v(z) \equiv \varphi'/\varphi$ .

After identifying  $\varphi$  with the dimensionless quantity  $\tilde{\varphi} \equiv \varphi/M_{pl}$ , the construction equations are of the same form as Eqs. (9) and (12), which relate the non-canonical kinetic term  $\tilde{p}(X)$  to the equation of state function  $w_k(z)$ . Given an effective equation of state function  $w_k(z)$ , Eq. (13) will make these two equations (9) and (12) simulate the non-canonical kinetic energy  $\tilde{p}(X)$ .

Our method relates directly the kinetic terms of  $k$ -essence to the equation of state function, and so enables us to construct easily the former without assuming its form. The dark energy properties are well described by the effective equation of state parameter  $w_k(z)$  which in general depends on the redshift  $z$ .

Following the previous work,<sup>34</sup> we consider the construction process with the following parametrization methods<sup>28–33</sup>: a constant equation of state parameter and the other three two-parameter parametrizations.

**Case 1.**  $w_k = w_0$  (Ref. 28)

**Case 2.**  $w_k = w_0 + w_1 z$  (Ref. 29)

**Case 3.**  $w_k = w_0 + w_1 z/(1+z)$  (Refs. 30–32)

**Case 4.**  $w_k = w_0 + w_1 \ln(1+z)$  (Ref. 33)

We choose  $w_0 = -0.7$ ,  $w_1 = -0.2$  and  $r_0 = 3/7$  for the specific construction.<sup>37</sup> This is quite different from the quintessence case as  $dw_k/dz > 0$  is generally required for the former.<sup>13,14,38</sup> This noticeable feature would enable supernova data to be distinguished between the two theories, although some ambiguity is difficult to deal with. Just for definiteness we have chosen the initial values of the  $k$ -essence field  $\varphi_0 = 1.0$  and  $v_0 = 0.1$  at  $z = 0$ . This degree of freedom of initial value settings will be easily eliminated after appropriate field rescaling and have no essential influence on the general form of  $\tilde{p}(X)$ . As a matter of fact, the change of the initial value  $\varphi_0$  or  $v_0$  just lifts the corresponding curves vertically and keeps their shapes intact. The numerical results are plotted as follows. Figure 1 shows the constructed non-canonical kinetic term  $\tilde{p}(X)$ . At low redshift region ( $z < 1$ , corresponding to  $X > 4$ ), all these models share the same asymptotical behavior, but begin to deviate from this around redshift  $z > 1$ .

We should note that the parametrizations we adopted here might not be of particular significance but just for concreteness. Since there is little theoretical guidance

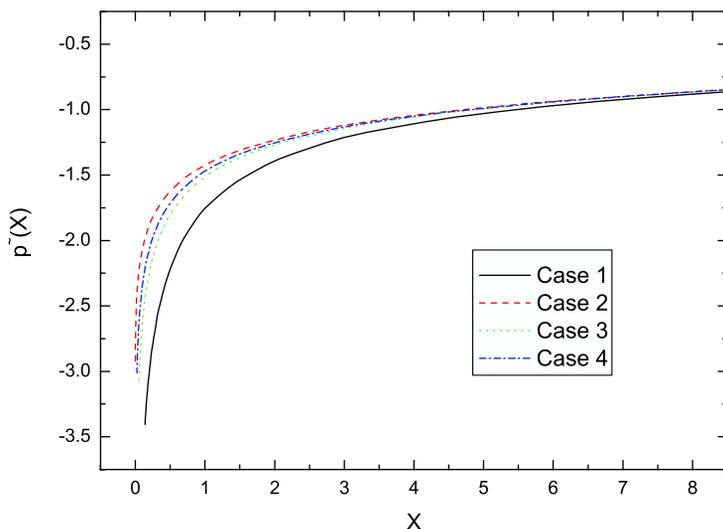


Fig. 1. Constructed kinetic term  $\tilde{p}(X)$  in the  $k$ -essence model.

as to the nature of the dark energy, the phenomenological parametrization of  $w_k(z)$  should be designed as generally as possible unless no observations can effectively unravel such excess complication. It has been shown that if there are more than two additional parameters, the system is not viable for a general fit with next generation data but two-parameter approximations in current use are reasonable and realistic for the near future observations.<sup>38–40</sup> On the other hand, due to the lack of data and the large statistical error bars, a wide range of dark energy models seem to have ranges of parameters that fit the existing data equally well and no particular model stands out as being observationally preferred. As a consequence, any such parametrizations made in potential reconstruction methods may be too restrictive since many different potential energy functions are conceivable and many of them may give results degenerate with each other. While it would still be difficult to break this observational degeneracy, the challenging task of making the fundamental physics distinction of the sign of the EOS time variation is achievable in the next generation of experiments.<sup>39,40</sup> One of the direct rewards relevant to the two-parameter approximation would then be that the usual form of  $k$ -essence with  $dw_k/dz > 0$  could be taken away. Anyhow, although we have concentrated specially on the  $k$ -essence form in Ref. 35, it is obvious that the techniques exploited here may be applied to more general  $k$ -essence models as well.

In conclusion, we have used the scheme of constructing the form of dark energy developed in Ref. 34 directly from the effective equation of state function  $w_k(z)$ . Then we have considered four parametrizations of equation of state parameter and showed that the constructed  $k$ -essence model may also be constructed without further dynamical information of the  $k$ -essence field. Current popular parametrizations

we have referred to are directly related to the data of type Ia supernovae; therefore, our approach will be a useful tool to re-examine the  $k$ -essence models, and the two parameters extracted from the SNe Ia observations will not only help us select the suitable  $k$ -essence form or rule out the usual form of  $k$ -essence models with great confidence in the future, but may also lead to a strong constraint and convenient guidance for the fundamental theory of the perplexing dark energy. The future Supernova/Acceleration Probe with high-redshift observations,<sup>a</sup> in combination with the Planck CMB observation,<sup>b</sup> will be able to determine the parameters in the dark energy parametrization to high precision. By precision mapping of the recent expansion history, we hope to learn more about the essence of the dark energy and get a deeper understanding of the dynamics of the universe.

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<sup>a</sup>SNAP homepage: <http://snap.lbl.gov>.

<sup>b</sup>Planck homepage: <http://www.rssd.esa.int/PLANCK>.

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